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NON-PLANAR CRACK WITH CRACK-FRONT PLASTIC YIELDING UNDER GENERAL LOADING

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ABSTRACT

The present study aims at providing expressions of the crack-tip stress and crack extension force when a non-planar crack of arbitrary shape with localized plastic yielding at its tips is loaded in mixed mode *I+II+III*. The considered model is a non-planar crack surrounded by a plastic zone with a dislocation-free zone DFZ in between, inside an infinitely extended elastic medium. The loadings, tension σ_{22}^{a} and shears σ_{12}^{a} and σ_{23}^{a}, are applied along the x_2, x_1 and x_3 directions at infinity, respectively. In addition, the treatment includes normal induced stresses which result from the Poisson's effect, acting perpendicularly to the direction of applied tension. The crack front is planar in x_2x_3, has an average elevation $h= h(x_1)$ from Ox_1x_3 and fluctuates weakly there in the form $\xi= \xi (x_1, x_3)$. The crack is represented by a continuous distribution of three types J (J = *I, II* and *III*) of infinitesimal dislocations having the shape of its front, with Burgers vectors (0, b, 0), (b, 0, 0) and (0, 0, b), respectively, directed along the applied loadings. The plastic region is on the average fracture plane and is also represented by these three types of dislocations with the exception that they are now straight dislocations parallel to x_3. Then distribution functions D_J of straight dislocation arrays, corresponding to an elastic-plastic crack π_0, inclined by an angle θ_0 with respect to Ox_1x_3, are calculated. Adopting these D_J, we propose explicit expressions of the crack-tip stresses and crack extension force per unit length of the crack front for a non-planar crack front of arbitrary shape. Except for a difference in the value of the stress intensity factor, that now incorporates the length of the DFZ and plastic zone, these quantities are identical with those of an isolated non-planar crack. This makes it possible to analyse brittle failure and fracture with crack-tip plasticity in the same way.

Keywords : *fracture mechanics, dislocation, crack-tip stress, energy release rate.*

RÉSUMÉ

Fissure non plane avec une zone plastique en tête de fissure sous sollicitation extérieure arbitraire

La présente étude fournit des expressions des contraintes en tête de fissure et force d'extension de fissure lorsqu'une fissure non plane de forme arbitraire, avec de la plasticité localisée à ses extrémités, est sollicitée en mode mixte *I+II+III*. Le modèle considéré est une fissure non plane entourée d'une région plastique entre lesquelles existe une zone libre de dislocation, à l'intérieur d'un milieu élastique infiniment étendu. La tension et les cisaillements sont appliqués le long des directions x_2, x_1 et x_3 à l'infini, respectivement. De plus, le traitement inclut des contraintes induites normales, résultant de l'effet de Poisson et agissant perpendiculairement à la direction de la tension appliquée. Le front de fissure est plan parallèle à x_2x_3, a une côte moyenne $h = h(x_1)$ par rapport à Ox_1x_3 et ondule faiblement sous la forme $\xi = \xi(x_1, x_3)$ à cette hauteur. La fissure est représentée par une distribution continue de trois types *J (J=I, II* et *III)* de dislocation ayant la forme du front de fissure. Leurs vecteurs de Burgers $\boldsymbol{b}_J$ sont dirigés le long de la tension et des cisaillements appliqués, respectivement. La région plastique est sur le plan de fracture moyen et est également représentée par ces trois types de dislocation, à l'exception qu'il s'agit maintenant de dislocations droites parallèles à x_3. Puis des fonctions de distribution D_J de dislocations droites, correspondant à une fissure élastique-plastique π_0, inclinée d'un angle θ_0 par rapport à Ox_1x_3, sont déterminées. En adoptant ces D_J, nous proposons des expressions explicites des contraintes en tête de fissure et de la force d'extension de fissure G (par unité de longueur du front de fissure), pour une fissure non plane de forme arbitraire. Exceptée une différence dans la valeur du facteur d'intensité de contrainte, qui intègre désormais les dimensions de la zone plastique et celle libre de dislocation, les grandeurs obtenues sont identiques à celles d'une fissure non plane isolée. Ce qui permet d'analyser de la même manière la rupture fragile et celle associée à de la plasticité localisée en tête de fissure.

Mots-clés : *mécanique de la rupture, dislocation, force d'extension de la fissure, contrainte en tête de fissure.*

I - INTRODUCTION

The spread of plastic yielding from a planar straight-fronted crack in a solid has been studied theoretically and experimentally [1 - 3]. One common description of cracking and plasticity in terms of dislocations has been introduced in the modelling [2] : both the crack and the plastic region are represented by continuous distributions of dislocations with infinitesimal Burgers vector *b*. On further modelling, a dislocation free zone (DFZ) between the crack and the physical dislocations has been included into the analysis [4 - 7]. This allows high stresses to be attained at the tip of the crack, a necessary condition for brittle fracture propagation, and common crack tip behaviors between isolated cracks and cracks surrounded by physically observable dislocations (using transmission electron microscopy, for example). Hence, there is one-one similarity in form (in both cases) of crack characteristic quantities : crack dislocation distribution D_0^J and corresponding relative displacement ϕ_0^J of the faces of the crack, crack-tip stress σ_0^J and crack extension force G_0^J. If the initiation of crack propagation occurs while the configuration of the plastic zone is fixed, it is evident that the crack extension force will be unchanged in form. This assumption is applicable to several experiments [8, 9]. However, this restriction is not mandatory. If no stress singularity exists at the boundary of the plastic zone, the crack extension force will remain unchanged. Considering in Ox_1x_3, a planar crack of finite extension $|x_1| \leq c$ along x_1 surrounded by plastic zone of extension $e \leq |x_1| \leq a$, ($e > c$), both running indefinitely along x_3 under mode of applied loading *J* (*J=I, II* and *III*), we have

$$D_0^J = \frac{K_J^0}{\pi\left[C_1\left(\delta_{JI}+\delta_{JII}\right)+\delta_{JIII}C_2\right]\sqrt{2\pi s_1}}, \quad \phi_0^J = \frac{2bK_J^0\sqrt{s_1}}{\pi\left[C_1\left(\delta_{JI}+\delta_{JII}\right)+\delta_{JIII}C_2\right]\sqrt{2\pi}},$$

$$\sigma_0^J = \frac{K_J^0}{\sqrt{2\pi s}}, \qquad G_0^J = \frac{1-\nu^2}{E}K_J^{02}\left(\delta_{JI}+\delta_{JII}\right)+\frac{1+\nu}{E}K_{III}^{0\ 2}\delta_{JIII}; \tag{1}$$

E, μ and ν are Young and shear modules and Poisson's ratio respectively; $C_1 = \mu b / 2\pi(1-\nu)$, $C_2 = \mu b / 2\pi$, $s_1 = c - x_1$ ($0 < s_1 << c$), $s = x_1 - c$ ($0 < s << c$); δ_{ij} is the Kronecker delta; K_J^0 is the stress intensity factor, it reads ([4 - 7]; see also below, in the present work)

$$K_J^0 = \left\{\sigma_J^a(0)\sqrt{\pi c}\right\}\frac{F(\pi/2,k)\sqrt{a^2-c^2}}{\sqrt{e^2-c^2}\ \Pi(\pi/2, e^2k^2/c^2, k)}, \qquad k^2 = \frac{c^2(a^2-e^2)}{e^2(a^2-c^2)}; \tag{2}$$

F and Π are elliptic integrals of first and third kind, respectively; $\sigma_J^a(0)$ takes the values σ_{22}^a ($J = I$), σ_{12}^a ($J = II$), and σ_{23}^a ($J = III$) corresponding to the remote applied tension along x_2 and shears parallel to x_1 and x_3, respectively. In K_J^0 (2), the factor in curly brackets { } is the stress intensity factor for an isolated crack of finite length $2c$. From above, it is convincing that brittle isolated crack and crack under crack-tip yielding can be analysed in the same way, equating G_0^J to twice the surface energy under pure mode J of applied loading. So far, only planar cracks are considered; however as stressed by [7], dislocation generation from the tip of the crack modifies the shape of the crack from planar to non-planar. Equally well, under mixed mode I+II+III loading, the applied shearing stresses promote non-planar crack motion (we may refer to [10,11] and references there in). In these cases, results like (1) and (2) now depend on the shape of the non-planar crack-front and orientation of the average fracture plane with respect of the applied loadings. The aim of the present study is to analyse a model of non-planar crack with associated DFZ and plastic domain under mixed mode I+II+III, on the similar lines, describing crack and plastic zone as continuous distributions of infinitesimal dislocations. In Section 2, the model and methodology of analysis is presented. Section 3 and 4 are devoted to results and discussion. Section 5 gives a conclusion to this study.

II - METHODOLOGY

The non-planar crack model and associated treatment are like those of [10, 11]. Along x_1, the crack extends from x_1 = - c to c with a x_2x_3-planar front. It is represented by a continuous distribution of three families J (J=I, II and III) of infinitesimal dislocations (types I and II are edges on average and type III screws). The identical shape f of the dislocations spreads in x_2x_3 and depends on x_1 and x_3 in the form

$$f = \sum_n \left(\xi_n \sin\kappa_n x_3 + \delta_n \cos\kappa_n x_3\right) + h(x_1) \equiv \xi(x_1, x_3) + h(x_1)\,. \tag{3}$$

Here n is a positive integer; h, ξ_n, δ_n and κ_n are real that are x_1-dependent. On both sides of the crack, a dislocation free zone $c \le |x_1| \le e$ and plastic region $e \le |x_1| \le a$ are present. In the plastic zone, the dislocations J are straight parallel to x_3 and cover the average fracture surface $x_2 = h(x_1)$. In both the crack and plastic region, the dislocation J (J=I, II and III) Burgers vectors are $\vec{b}_I = (0, b, 0)$, $\vec{b}_{II} = (b, 0, 0)$ and $\vec{b}_{III} = (0, 0, b)$, respectively. The medium is infinite, isotropic and elastic, subjected at infinity to uniform applied tension σ_{22}^a and shears σ_{12}^a and σ_{23}^a. In addition, the treatment includes normal induced stresses $\sigma_{11}^a = -\nu_A \sigma_{22}^a$ and $\sigma_{33}^a = -\nu_A \sigma_{22}^a$ ($\nu_A = \nu$). The notation ν_A permits to identify in the various listed mathematical expressions below, those associated with these induced normal stresses. The dislocation distribution functions D_J

(*J*= *I* , *II* and *III*) are defined such that $D_J(x_1^{'})dx_1^{'}$ represents the number of dislocations *J* in the infinitesimal x_1 – interval $dx_1^{'}$ located about the x_1 – spatial position $x_1^{'}$. Let ($\bar{\sigma}$) be the total stress at any spatial position *P* (x_1, x_2, x_3) in the medium. It can be written

$$\bar{\sigma}_{ij} = \sigma_{ij}^{A} - \sigma_{ij}^{fr} + \sum_{J=I}^{III} \bar{\sigma}_{ij}^{(J)} \tag{4}$$

$$(\sigma)^A = \begin{pmatrix} -\nu_A \sigma_{22}^a & \sigma_{12}^a & 0 \\ \sigma_{12}^a & \sigma_{22}^a & \sigma_{23}^a \\ 0 & \sigma_{23}^a & -\nu_A \sigma_{22}^a \end{pmatrix} \tag{5}$$

$(\sigma)^{fr}$ is the friction stress opposing the motion of the dislocations in the plastic region. We assume that no friction stress acts on the crack dislocations (i.e. these are slipping and climbing freely).

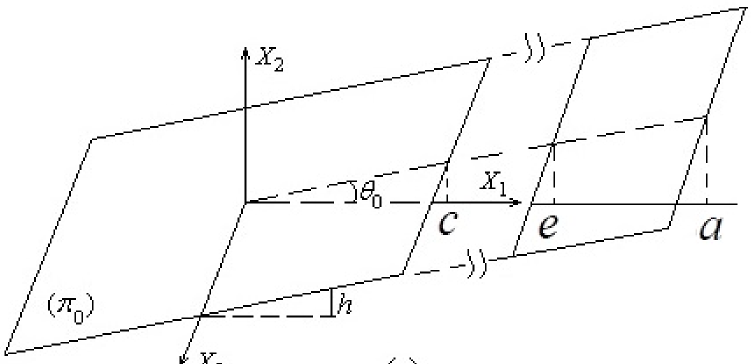
$$\bar{\sigma}_{ij}^{(J)} = \left(\int_{-a}^{-e} + \int_{-c}^{c} + \int_{e}^{a} \right) \sigma_{ij}^{(J)}(x_1 - x_1^{'}, x_2, x_3) D_J(x_1^{'}) dx_1^{'} \tag{6}$$

$\sigma_{ij}^{(J)}$ is the stress due to the dislocation *J* located at spatial position $x_1 = x_1^{'}$ in the distribution. We are concerned with finding the equilibrium distribution D_J of the dislocations under the combined action of their mutual repulsion, the force exerted on them by the applied stresses and the friction stress. The equilibrium condition is equivalent to asking that at any spatial position on the crack and plastic zone, the tractions are zero:

$$\begin{cases} \bar{\sigma}_{12} - \partial f / \partial x_1 \bar{\sigma}_{11} - \partial f / \partial x_3 \bar{\sigma}_{13} = 0 \\ \bar{\sigma}_{22} - \partial f / \partial x_1 \bar{\sigma}_{12} - \partial f / \partial x_3 \bar{\sigma}_{23} = 0 \\ \bar{\sigma}_{23} - \partial f / \partial x_1 \bar{\sigma}_{13} - \partial f / \partial x_3 \bar{\sigma}_{33} = 0 \end{cases} . \tag{7}$$

Additional requirement is that the stresses at the boundary of the plastic zone are bounded. This corresponds to $D_J(\pm e) = D_J(\pm a) = 0$. When D_J from (7) is known, the relative displacement of the faces of the crack, the crack-tip stress and crack extension force are derived by integration. In its general form, (7) requires a numerical resolution. Fortunately, as performed below, approximate expressions for the crack-tip stresses and crack extension force with f (3) can be given, taking for D_J those of straight dislocation arrangements obtained in Section 3 below; this is the usual procedure in our works [10, 11]. ***Figure 1*** and 2 are schematical representations of simple special elastic – plastic cracks captured by the modelling. The crack and plastic zone cover the x_1- intervals $|x_1| \le c$ and $e \le |x_1| \le a$ with DFZ in between; these must be considered running indefinitely in the x_3- direction. The crack shape in planes perpendicular to x_1 is described by ξ (***Figure 2a*** for example). The shape f of the crack in planes perpendicular to x_3 is given by both ξ, through the x_1 – dependence of positive quantities ξ_n, δ_n and κ_n (Equation (3)), and function $h = h(x_1)$. Since ξ is assumed to be small oscillating function, the average fracture surface is described correctly by the equation $x_2 = h(x_1)$. When $\xi = 0$, the crack dislocations are straight parallel to x_3 and distributed over the surface $x_2 = h(x_1)$. The plastic region dislocations are straight parallel to x_3 and cover the average fracture surface $x_2 = h(x_1)$ Specific examples are given in ***Figure 1 and 2*** where the plastic region for positive x_1 only is shown :

- $h(x_1) = p_0 x_1$ ($p_0 \ge 0$) and $\xi = 0$. This corresponds to a planar elastic – plastic crack π_0 (with a straight front parallel to x_3) rotated around Ox_3 by angle $\theta_0 = \tan^{-1} p_0$ from Ox_1x_3, ***Figure 1a***.
- $h(x_1)$ is an arbitrary function of x_1, $\xi = 0$. The sketch in ***Figure 1b*** corresponds to h odd although this is not mandatory. Actually h odd conforms well to homogeneity of the medium, geometry of the applied loadings and D_J (Section 3) approximation adopted in the present study.



(a)

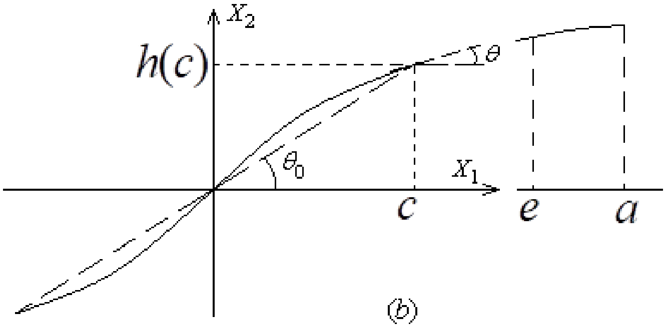
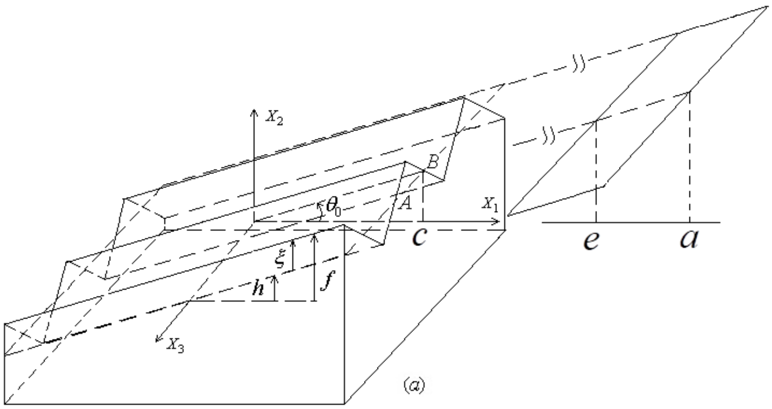


Figure 1 : *Simple special elastic – plastic cracks. (a) Inclined planar elastic – plastic crack π_0 (see text). (b) A non-planar elastic – plastic crack (parallel to x_3) as h odd function of x_1 ($x_2 = h(x_1)$)*

- $h(x_1) = p_0 x_1$ ($p_0 \geq 0$) and $\xi = \xi(x_3)$ independent of x_1. The crack fluctuates about plane π_0 with a front spreading in planes parallel to $x_2 x_3$ in the form ξ. In the example displayed in ***Figure 2a*** the crack consists of planar facets with inclination angles ϕ_A and ϕ_B ***(Figure 2b)*** at points *A* and *B* of the crack front located on the average fracture plane. Points *A* and *B* are indicated in ***Figure 2a***.



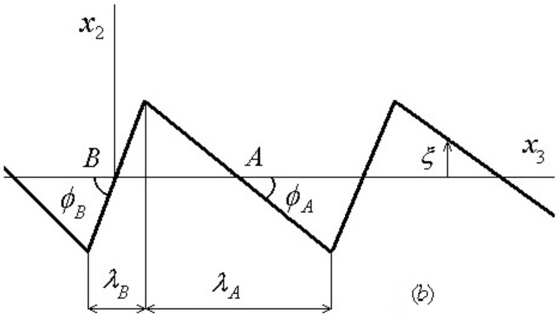


Figure 2 : *Special elastic – plastic crack. (a) Non-planar elastic crack fluctuating about an average inclined plane π_0. The crack consists of planar facets; its fronts at $x_1 = \pm c$ lie in x_2x_3 – planes. At $x_1 = c$, the crack front is characterized by inclination angles ϕ_A and ϕ_B (see (b)) at points A and B located on the average fracture plane. The plastic zone lies on π_0. (b) Sketch of the crack front in (a) with B taken as origin*

III - RESULTS

III-1. Dislocation distributions

Assume first that the dislocations are straight parallel to the x_3 – direction $(\xi = 0)$ and $h(x_1) = p_0 x_1$ depends linearly on x_1 with p_0 positive constant. This corresponds to a planar elastic – plastic crack (π_0 in ***Figure 1a***) of finite extension (with straight fronts running indefinitely along x_3) rotated about the positive x_3 – direction by $\theta_0 = \tan^{-1} p_0$ from Ox_1x_3. The elastic – plastic crack ($|x_1| \le c$, $e \le |x_1| \le a$) is subjected to mixed mode $I+II+III$ with loadings applied at infinity. Under such conditions $\partial f / \partial x_1 = \partial h / \partial x_1 = p_0$ and $\partial f / \partial x_3 = \partial \xi / \partial x_3 = 0$; the condition (7) for the elastic – plastic crack faces to be free from tractions becomes:

$$\sigma_J + \left[C_1(\delta_{JI} + \delta_{JII}) + C_2 \delta_{JIII} \right] \left(\int_{-a}^{-e} + \int_{-c}^{c} + \int_{e}^{a} \right) \frac{D_J(x_1^{'})}{x_1 - x_1^{'}} dx_1^{'} = 0 \quad (J = I, II \text{ and } III). \qquad (8)$$

For $|x_1| \le c$

$$\sigma_I = \sigma_{22}^a - p_0 \sigma_{12}^a \equiv \sigma_I^a, \; \sigma_{II} = \sigma_{12}^a + p_0 \nu_A \sigma_{22}^a \equiv \sigma_{II}^a \text{ and } \sigma_{III} = \sigma_{23}^a \equiv \sigma_{III}^a$$

For $e \le |x_1| \le a$

$\sigma_I = \sigma_I^a - (\sigma_{22}^{fr} - p_0\sigma_{12}^{fr}) \equiv \sigma_I^a - \sigma_I^{fr}$, $\sigma_{II} = \sigma_{II}^a - (\sigma_{12}^{fr} - p_0\sigma_{11}^{fr}) \equiv \sigma_{II}^a - \sigma_{II}^{fr}$ and $\sigma_{III} = \sigma_{III}^a - \sigma_{23}^{fr} \equiv \sigma_{III}^a - \sigma_{III}^{fr}$

The solution to (8) is known in terms of the Jacobi's zeta function *Z (φ, k)* [7]. For $|x_1| < c$

$$D_J(x_1) \equiv D_J^C(x_1) = \frac{2\sigma_J^{fr} F(\pi/2, k)}{\pi^2\left[C_1(\delta_{JI} + \delta_{JII}) + C_2\delta_{JIII}\right]}\left(\frac{(e^2 - c^2)x_1}{e\sqrt{a^2 - c^2}}\right.$$

$$\left.\times\sqrt{\frac{a^2 - x_1^2}{(c^2 - x_1^2)(e^2 - x_1^2)}} + \operatorname{sgn}(x_1) Z\left(\beta_1(x_1), k\right)\right) \quad (9)$$

For $e \le |x_1| \le a$

$$D_J(x_1) \equiv D_J^{ea}(x_1) = \frac{2\sigma_J^{fr} F(\pi/2, k)}{\pi^2\left[C_1(\delta_{JI} + \delta_{JII}) + C_2\delta_{JIII}\right]} \operatorname{sgn}(x_1) Z\left(\beta_2(x_1), k\right) \quad (10)$$

$$\beta_1(x_1) = \sin^{-1}\left(1/\sqrt{n(x_1)}\right),\ \beta_2(x_1) = \sin^{-1}\left(\sqrt{n(x_1)}/k\right),\ k^2 = \frac{c^2(a^2 - e^2)}{e^2(a^2 - c^2)},$$

$$n(x_1) = \frac{c^2(e^2 - x_1^2)}{e^2(c^2 - x_1^2)}.$$

It must be attached to this solution of D_J [7] the following relation between the applied stresses, friction stresses, and size of the elastic – plastic crack

$$\frac{\pi\sigma_J^a}{2\sigma_J^{fr}} = \frac{e^2 - c^2}{e\sqrt{a^2 - c^2}} \Pi\left(\pi/2, e^2k^2/c^2, k\right). \quad (11)$$

Because $\sigma_J^a = \sigma_J^a(p_0)$ and $\sigma_J^{fr} = \sigma_J^{fr}(p_0)$ (see about (8)) depend on p_0, relation (11) is valid for all p_0; this leads to $\sigma_J^a(p_0)/\sigma_J^{fr}(p_0) = \sigma_J^a(0)/\sigma_J^{fr}(0)$ constant with p_0. It can thus be written

$$D_J(x_1) = \frac{\sigma_J^a(p_0)}{\sigma_J^a(0)} D_0^J(x_1) \quad (12)$$

Where D_0^J is the value of D_J (9, 10) at $p_0 = 0$; D_0^J (J= *I*, *II* and *III*) corresponds to the equilibrium distribution of straight dislocations *J* when the elastic – plastic crack is planar in Ox_1x_3.

$$\frac{\sigma_I^a(p_0)}{\sigma_I^a(0)} = 1 - p_0 \frac{\sigma_{12}^a}{\sigma_{22}^a}, \quad \frac{\sigma_{II}^a(p_0)}{\sigma_{II}^a(0)} = 1 + p_0 \frac{\nu_A \sigma_{22}^a}{\sigma_{12}^a}, \quad \frac{\sigma_{III}^a(p_0)}{\sigma_{III}^a(0)} = 1. \tag{13}$$

The relative displacement ϕ_J^C of the faces of the crack, in the x_2 ($J = I$), x_1 ($J = II$) and x_3 ($J = III$) directions reads

$$\phi_J^C(x_1) = \int_{x_1}^{c} bD_J^C(x_1^{'})dx_1^{'}, \qquad |x_1| \le c,$$

which gives after integration

$$\phi_J^C(x_1) = \frac{\sigma_J^a(p_0)}{\sigma_J^a(0)} \phi_0^{CJ}(x_1) \tag{14}$$

where

ϕ_0^{CJ} is the value of ϕ_J^C at $p_0 = 0$ (see relation (16) in [7] for $J = I$). We have

$$\begin{aligned}\phi_0^{CJ} = \frac{2b\sigma_J^{fr}(0)}{\pi^2\left[C_1(\delta_{JI} + \delta_{JII}) + C_2\delta_{JIII}\right]} \Bigg(& \frac{F(\pi/2,k)(e^2 - c^2)}{e\sqrt{a^2 - c^2}} \sqrt{\frac{(a^2 - x_1^2)(c^2 - x_1^2)}{e^2 - x_1^2}} \\ & - F\left(\frac{\pi}{2},k\right)|x_1| Z\left(\beta_1(x_1),k\right) - ek^2\left[F\left(\frac{\pi}{2},k\right)\hat{D}\left(\beta_3(x_1),\frac{ek}{c}\right)\right. \\ & \left.\left. - F\left(\beta_3(x_1),\frac{ek}{c}\right)\hat{D}\left(\frac{\pi}{2},k\right)\right]\right), \quad |x_1| \le c\end{aligned} \tag{15}$$

$\hat{D}$ is a function of variables (φ, k) defined as $\hat{D}(\varphi,k) = \left(F(\varphi,k) - E(\varphi,k)\right)/k^2$ in which E is the elliptic integral of the second kind, $\beta_3(x_1) = \sin^{-1}\left(c/e\sqrt{n(x_1)}\right)$. The number of dislocations J in the plastic zone between e and x_1 is

$$N_J^{ea}(x_1) = \frac{\sigma_J^a(p_0)}{\sigma_J^a(0)} N_0^{ea(J)}(x_1) \tag{16}$$

where $N_0^{ea(J)}$ is the value of N_J^{ea} for $p_0 = 0$.

$$N_0^{ea(J)} = \frac{2\sigma_J^{fr}(0)}{\pi^2\left[C_1(\delta_{JI} + \delta_{JII}) + C_2\delta_{JIII}\right]} \left(F\left(\frac{\pi}{2},k\right) x_1 Z\left(\beta_2(x_1),k\right)\right.$$

$$+ek^2\left(F\left(\frac{\pi}{2},k\right)\hat{D}\left(\beta_2(x_1),\frac{ek}{c}\right)-F\left(\beta_2(x_1),\frac{ek}{c}\right)\hat{D}\left(\frac{\pi}{2},k\right)\right)\Bigg),\ e\le x_1\le a. \quad (17)$$

The total number $\bar{N}_J^{ea}$ of dislocations J between e and a is thus

$$\begin{aligned}\bar{N}_J^{ea} &= \frac{\sigma_J^a(p_0)}{\sigma_J^a(0)}\frac{2e\sigma_J^{fr}(0)k^2}{\pi^2\left[C_1(\delta_{JI}+\delta_{JII})+C_2\delta_{JIII}\right]}\left(F(\frac{\pi}{2},k)\hat{D}(\frac{\pi}{2},\frac{ek}{c})-F(\frac{\pi}{2},\frac{ek}{c})\hat{D}(\frac{\pi}{2},k)\right)\\ &= \frac{\sigma_J^a(p_0)(a^2-e^2)c^2}{\pi\left[C_1(\delta_{JI}+\delta_{JII})+C_2\delta_{JIII}\right]}\left(F(\frac{\pi}{2},k)\hat{D}(\frac{\pi}{2},\frac{ek}{c})-F(\frac{\pi}{2},\frac{ek}{c})\hat{D}(\frac{\pi}{2},k)\right)\\ &\quad\times\frac{1}{(e^2-c^2)\sqrt{a^2-c^2}\,\Pi(\pi/2,e^2k^2/c^2,k)}\end{aligned} \quad (18)$$

As performed below, we can reasonably give approximate expressions for the stress about the crack front in the DFZ and crack extension force with f given by (3) when the average fracture surface h can be approximated by plane π_0 of ***Figure 1a***.

III-2. Stresses about the crack front and crack extension force

We would like to express the total stress $\bar{\sigma}_{ij}$ (4), ahead of the front of the crack with shape f (3), at $x_1 = c$ in the DFZ. Writing $x_1 = c + s$, $0 < s << c$, $\bar{\sigma}_{ij}$ is given by the following formula

$$\bar{\sigma}_{ij}(s,x_2,x_3)=\sum_{J=I}^{III}\int_{c-\delta c}^{c}\sigma_{ij}^{(J)}(c+s-x_1',x_2,x_3)D_J(x_1')dx_1' \quad (19)$$

with $\delta c << c$ and x_2 close to $h(c)$. Then, proceeding exactly as in our previous works [10, 11] taking for D_J the straight edge and screw dislocation distributions (12) corresponding to an elastic – plastic planar crack π_0 with a straight front parallel to x_3 ***(Figure 1a)***, we obtain ($\bar{\sigma}_{ij}\equiv\bar{\sigma}_{ij}^{(I)}+\bar{\sigma}_{ij}^{(II)}+\bar{\sigma}_{ij}^{(III)}$ (6)):

$$\begin{aligned}\bar{\sigma}_{ii}^{(I)}(s,x_2,x_3) &= \frac{1}{(1+p^2)^3}\Bigg([\delta_{i1}+\delta_{i2}+2\nu\delta_{i3}+(-\delta_{i1}+3\delta_{i2}+2\nu\delta_{i3})p^2](1+p^2)\\ &\quad+\frac{1}{2}\left(x_2-h(c)\right)\Big[\delta_{i1}-\delta_{i2}+2(1+\nu)\delta_{i3}-6(\delta_{i1}-\delta_{i2})p^2\\ &\quad-\left(-\delta_{i1}+\delta_{i2}+2(1+\nu)\delta_{i3}\right)p^4\Big]\frac{\partial^2\xi}{\partial x_3^2}(c,x_3)\Bigg)\left(1-p_0\frac{\sigma_{12}^a}{\sigma_{22}^a}\right)\frac{K_I^0}{\sqrt{2\pi}}\frac{1}{\sqrt{s}},\end{aligned}$$

$$\bar{\sigma}_{ii}^{(II)}(s,x_2,x_3)=\frac{(-\delta_{i1}+\delta_{i2}-\delta_{i3})p}{(1+p^2)^3}\Big([3\delta_{i1}+\delta_{i2}+2\nu\delta_{i3}+(\delta_{i1}-\delta_{i2}+2\nu\delta_{i3})p^2](1+p^2)$$
$$+\frac{1}{2}\left(x_2-h(c)\right)\left[3\delta_{i1}+(3+4\nu)\delta_{i2}+2(1+3\nu)\delta_{i3}+(-6\delta_{i1}-2(3-4\nu)\delta_{i2}+8\nu\delta_{i3})p^2\right.$$
$$\left.+\left(-\delta_{i1}-(1-4\nu)\delta_{i2}-2(1-\nu)\delta_{i3}\right)p^4\right]\frac{\partial^2\xi}{\partial x_3^2}(c,x_3)\Big)\left(1+p_0\frac{\nu_A\sigma_{22}^a}{\sigma_{12}^a}\right)\frac{K_{II}^0}{\sqrt{2\pi}}\frac{1}{\sqrt{s}},$$

$$\bar{\sigma}_{ii}^{(III)}(s,x_2,x_3)=\frac{\left[(p^2-1)\delta_{i1}+\left(1-2\nu-(1+2\nu)p^2\right)\delta_{i2}-2(1+p^2)\delta_{i3}\right]}{(1-\nu)(1+p^2)^2}\frac{\partial\xi}{\partial x_3}\frac{K_{III}^0}{\sqrt{2\pi}}\frac{1}{\sqrt{s}},$$

$$\bar{\sigma}_{j3}^{(I)}(s,x_2,x_3)=\frac{p[-(2-\nu)+\nu p^2]\delta_{j1}+[\nu-(2-\nu)p^2]\delta_{j2}}{(1+p^2)^2}\frac{\partial\xi}{\partial x_3}\left(1-p_0\frac{\sigma_{12}^a}{\sigma_{22}^a}\right)\frac{K_I^0}{\sqrt{2\pi}}\frac{1}{\sqrt{s}},$$

$$\bar{\sigma}_{j3}^{(II)}(s,x_2,x_3)=-\frac{[1-(1-\nu)p^2-(2-\nu)p^4]\delta_{j1}+p[1+2\nu-(1-2\nu)p^2]\delta_{j2}}{(1+p^2)^2}$$
$$\times\frac{\partial\xi}{\partial x_3}\left(1+p_0\frac{\nu_A\sigma_{22}^a}{\sigma_{12}^a}\right)\frac{K_{II}^0}{\sqrt{2\pi}}\frac{1}{\sqrt{s}},$$

$$\bar{\sigma}_{j3}^{(III)}(s,x_2,x_3)=\frac{1}{(1-\nu)(1+p^2)^2}\Big((1-\nu)[-p\delta_{j1}+\delta_{j2}](1+p^2)-\frac{1}{2}\left(x_2-h(c)\right)$$
$$\times\left[p\left(5-3\nu-(1+\nu)p^2\right)\delta_{j1}-\left(3-\nu-(3-\nu)p^2\right)\delta_{j2}\right]\frac{\partial^2\xi}{\partial x_3^2}\Big)\frac{K_{III}^0}{\sqrt{2\pi}}\frac{1}{\sqrt{s}}$$

,

$$\bar{\sigma}_{12}^{(I)}(s,x_2,x_3)=\frac{p}{(1+p^2)^3}\Big(1-p^4+\frac{1}{2}\left(x_2-h(c)\right)$$
$$\times\left[5-2\nu-2(1+2\nu)p^2+(1-2\nu)p^4\right]\frac{\partial^2\xi}{\partial x_3^2}\Big)\left(1-p_0\frac{\sigma_{12}^a}{\sigma_{22}^a}\right)\frac{K_I^0}{\sqrt{2\pi}}\frac{1}{\sqrt{s}},$$

$$\bar{\sigma}_{12}^{(II)}=\frac{1}{(1+p^2)^3}\left(1-p^4+\frac{1}{2}\left(x_2-h(c)\right)\left[1+2(13+2\nu)p^2+p^4\right]\frac{\partial^2\xi}{\partial x_3^2}\right)$$
$$\times\left(1+p_0\frac{\nu_A\sigma_{22}^a}{\sigma_{12}^a}\right)\frac{K_{II}^0}{\sqrt{2\pi}}\frac{1}{\sqrt{s}},$$

$$\bar{\sigma}_{12}^{(III)}(s,x_2,x_3)=-\frac{p(2-\nu-\nu p^2)}{(1-\nu)(1+p^2)^2}\frac{\partial\xi}{\partial x_3}\frac{K_{III}^0}{\sqrt{2\pi}}\frac{1}{\sqrt{s}} \qquad (20)$$

where subscripts i and j take the values (1, 2 and 3) and (1 and 2) respectively; $p=\partial h(c)/\partial x_1$, K_J^0 ($J=I, II$ and III respectively) is given by (2). It is stressed again that s, x_2 and x_3 are arbitrary, $s=x_1-c<<c$ ($s>0$) and $(x_2-h(c))$ is small. The parameter p_0 in (20) originates from a planar crack π_0 ***(Figure 1a)*** hypothetically assumed to approximate the average fracture surface $x_2 = h(x_1)$. One observes that (20) and the relation (10) in [11] are identical. The crack extension force G (per unit length of the crack front) is calculated in the same way as for the isolated non-planar crack [10, 11]. We define a reduced crack extension force $\tilde{G}$ as $\tilde{G} = G/(G_0^I + G_0^{II} + G_0^{III})$ with G_0^J given in (1) and obtain at $P_0(c, x_2 = f, x_3)$ (with $M_{12} \equiv \sigma_{12}^a / \sigma_{22}^a$, $M_{13} \equiv \sigma_{23}^a / \sigma_{22}^a$, $M_{23} \equiv \sigma_{23}^a / \sigma_{12}^a$)

$$\tilde{G}(P_0) = \sum_{i,j=1}^{3} \tilde{G}_j^{(i)}(P_0) \tag{21}$$

where

$$\tilde{G}_1^{(1)} = -\frac{1}{2(1+p^2)^3} \frac{\partial f(c,x_3)/\partial x_1}{\sqrt{1+(\partial f/\partial x_1)^2 + (\partial f/\partial x_3)^2}} \Bigg(2(1-p^4)(1-p_0 M_{12}) - 2p(1+p^2)$$
$$\times (3+p^2)(p_0 \nu_A + M_{12}) + \frac{2}{1-\nu}(p^4-1) M_{13} \frac{\partial \xi}{\partial x_3} + \Big[(1-6p^2+p^4)(1-p_0 M_{12})$$
$$- p(3-6p^2-p^4)\Big)(p_0 \nu_A + M_{12})\Big] \xi \frac{\partial^2 \xi}{\partial x_3^2}\Bigg) \frac{(1-\nu)(p_0 \nu_A + M_{12})}{\left[1-\nu+(1-\nu)M_{12}^2 + M_{13}^2\right]},$$

$$\tilde{G}_2^{(1)} = \frac{1}{2(1+p^2)^3} \frac{1}{\sqrt{1+(\partial f/\partial x_1)^2 + (\partial f/\partial x_3)^2}} \Bigg(2(1-p^4)\left[p(1-p_0 M_{12}) + M_{12} + p_0 \nu_A\right]$$
$$- \frac{2}{1-\nu} p(1+p^2)(2-\nu-\nu p^2) M_{13} \frac{\partial \xi}{\partial x_3} + \Big[p\Big(5-2\nu-2(1+2\nu)p^2 + (1-2\nu)p^4\Big)$$
$$\times (1-p_0 M_{12}) + \Big(1+2(13+2\nu)p^2 + p^4\Big)(M_{12} + p_0 \nu_A)\Big]$$

$$\times \xi \frac{\partial^2 \xi}{\partial x_3^2}\Bigg) \frac{(1-\nu)(p_0 \nu_A + M_{12})}{\left[1-\nu+(1-\nu)M_{12}^2 + M_{13}^2\right]},$$

$$\tilde{G}_3^{(1)} = -\frac{1}{(1+p^2)^2} \frac{\partial f/\partial x_3}{\sqrt{1+(\partial f/\partial x_1)^2 + (\partial f/\partial x_3)^2}} \Bigg(-p(1+p^2) M_{13} + \Big[p\Big(-2+\nu+\nu p^2\Big)$$
$$\times (1-p_0 M_{12}) - \Big(1-(1-\nu)p^2 - (2-\nu)p^4\Big)(p_0 \nu_A + M_{12})\Big] \frac{\partial \xi}{\partial x_3}$$
$$- p\Big(5-3\nu-(1+\nu)p^2\Big) \frac{M_{13}}{2(1-\nu)} \xi \frac{\partial^2 \xi}{\partial x_3^2}\Bigg) \frac{(1-\nu)(p_0 \nu_A + M_{12})}{\left[1-\nu+(1-\nu)M_{12}^2 + M_{13}^2\right]},$$

$$\tilde{G}_1^{(2)} = -\tilde{G}_2^{(1)} \frac{\partial f}{\partial x_1} \frac{1 - p_0 M_{12}}{p_0 \nu_A + M_{12}},$$

$$\tilde{G}_2^{(2)} = \frac{1}{2(1+p^2)^3} \frac{1}{\sqrt{1+(\partial f/\partial x_1)^2 + (\partial f/\partial x_3)^2}} \Big(2(1+p^2)\big[(1+3p^2)(1-p_0 M_{12})$$
$$+ p(1-p^2)(M_{12}+p_0\nu_A)\big] + \frac{2}{1-\nu}(1+p^2)(1-2\nu-(1+2\nu)p^2)M_{13}\frac{\partial \xi}{\partial x_3}$$
$$+ \big[(-1+6p^2-p^4)(1-p_0M_{12}) + p\big(3+4\nu-2(3-4\nu)p^2-(1-4\nu)p^4\big)(M_{12}+p_0\nu_A)\big]$$
$$\times \xi \frac{\partial^2 \xi}{\partial x_3^2}\Big) \frac{(1-\nu)(1-p_0M_{12})}{\left[1-\nu+(1-\nu)M_{12}^2+M_{13}^2\right]},$$

$$\tilde{G}_3^{(2)} = -\frac{1}{(1+p^2)^2} \frac{\partial f/\partial x_3}{\sqrt{1+(\partial f/\partial x_1)^2 + (\partial f/\partial x_3)^2}} \Big((1+p^2)M_{13} + \Big[\Big(\nu-(2-\nu)p^2\Big)$$
$$\times (1-p_0M_{12}) - p\big(1+2\nu-(1-2\nu)p^2\big)(M_{12}+p_0\nu_A)\Big]\frac{\partial \xi}{\partial x_3} + \frac{(3-\nu)(1-p^2)M_{13}}{2(1-\nu)}$$
$$\times \xi \frac{\partial^2 \xi}{\partial x_3^2}\Big) \frac{(1-\nu)(1-p_0M_{12})}{\left[1-\nu+(1-\nu)M_{12}^2+M_{13}^2\right]},$$

$$\tilde{G}_1^{(3)} = \tilde{G}_3^{(1)} \frac{\partial f/\partial x_1}{\partial f/\partial x_3} \frac{M_{13}}{(1-\nu)(p_0\nu_A + M_{12})},$$

$$\tilde{G}_2^{(3)} = -\tilde{G}_3^{(2)} \frac{1}{\partial f/\partial x_3} \frac{M_{13}}{(1-\nu)(1-p_0M_{12})},$$

$$\tilde{G}_3^{(3)} = -\frac{1}{2(1+p^2)^3} \frac{\partial f/\partial x_3}{\sqrt{1+(\partial f/\partial x_1)^2 + (\partial f/\partial x_3)^2}} \Big(4\nu(1+p^2)^2\left[1-(p+p_0)M_{12} - pp_0\nu_A\right]$$
$$- \frac{4}{1-\nu}(1+p^2)^2 M_{13}\frac{\partial \xi}{\partial x_3} + \big[2(1+\nu)(1-p^4)(1-p_0M_{12})$$
$$- p\big(2(1+3\nu)+8\nu p^2 - 2(1-\nu)p^4\big)(M_{12}+p_0\nu_A)\big]\xi\frac{\partial^2 \xi}{\partial x_3^2}\Big) \frac{M_{13}}{\left[1-\nu+(1-\nu)M_{12}^2+M_{13}^2\right]}. \quad (22)$$

(22) is identical to the corresponding one for the isolated crack (see (16) in [11]). Expression (22) gives $\tilde{G}$ for any arbitrary shape f (3) of the crack front. Since fracture over large distance proceeds through the motion of a macroscopic length of the crack front, the relevant quantity is $<\tilde{G}>$, the value of $\tilde{G}$ averaged over the length of the crack front. $<\tilde{G}>$ for several special isolated crack fronts have been provided: straight, sinusoidal, segmented [10 - 13]. All these results apply here i.e. when the crack front is associated with localized plastic yielding.

The expected crack configuration after fracture propagation over large distance is the one that maximizes $<\tilde{G}>$ under the Griffith condition $<G>_{max} = 2\gamma$ where γ is the surface energy.

IV - DISCUSSION

The reduced crack extension force $\tilde{G}$ (22) for the elastic – plastic crack is identical to that of the isolated crack given in [11]. Consequently the various graphical representations of $<\tilde{G}>$ in [11], as a function of parameters (p, p_0, p_A, p_B; M_{12}, M_{13}; ν_A), for simple special isolated cracks are unchanged for the elastic-plastic cracks (see ***Figure 1 and 2***, for example). From these works, non-planar crack configurations exist for which the crack extension force $< G >$ is larger than that of the planar crack in Ox_1x_3, thus corroborating the occurrence of non-planar fracture abundantly observed in real materials. We shall add one more observation on the value of $<\tilde{G}>$ on inverting the sign of the applied shears σ_{12}^a and σ_{23}^a. We first consider the planar crack π_0 ***(Figure 1a)*** under mode I+II loading ($\sigma_{23}^a = 0$). We have

$$\tilde{G}(P_0) = \frac{1 + p_0^2\nu_A^2 + (1+p_0^2)M_{12}^2 - 2(1-\nu_A)p_0 M_{12}}{\sqrt{1+p_0^2}\left(1+M_{12}^2\right)} \quad (\xi = 0; h = p_0 x_1). \tag{23}$$

Inverting the sign of σ_{12}^a from positive to negative (i.e. $M_{12} < 0$) increases $\tilde{G}$. $M_{12} < 0$ corresponds to applying the shear in the negative x_1- direction when the specimen is suffering tension ($\sigma_{22}^a > 0$) along x_2. If the specimen is under fatigue by inverting the sign of σ_{12}^a only, this asymmetry implies that this is the condition $G = 2\gamma$ ($M_{12} > 0$) that controls the complete failure of the fracture material. A similar behaviour is found with σ_{23}^a for the non-planar crack with a segmented crack front ***(Figure 2a)***. We have in tension under mixed mode I+III ($\sigma_{12}^a = 0$) (see also (20) in [11]) :

$$<\tilde{G}>(\sigma_{23}^a < 0) - <\tilde{G}>(\sigma_{23}^a > 0) = \frac{2\nu v_1 |M_{13}|\left(2 + 4p_0^2 + (2-\nu)p_0^4 + (2-\nu)p_0^6\right)}{(1+p_0^2)^2\left(1-\nu+M_{13}^2\right)} \tag{24}$$

$$v_1 = \left(p_A p_B / (p_A + p_B)\right)\left(1/\sqrt{1+p_0^2+p_B^2} - 1/\sqrt{1+p_0^2+p_A^2}\right)$$

$p_A = \tan\phi_A$, $p_B = \tan\phi_B$ (see ***Figure 2b*** for ϕ_A and ϕ_B). If $p_A = p_B$, $v_1 = 0$ and there is no asymmetry on inverting the sign of the shear σ_{23}^{a}. However if $p_A \neq p_B$, $v_1 \neq 0$ and an asymmetry does exist. The controlling $<\tilde{G}>$ will be the smallest one depending on the values of p_A compared with p_B. Similarly, one may invert the sign of σ_{22}^{a} (< 0) and maintains the shearing stresses positive. Under such conditions (23) and (24) are unchanged; this leads to same asymmetries (note that σ_{22}^{a} replaces σ_{23}^{a} in the left hand side in (24)). One notes that Poisson's stress $\sigma_{11}^{a} = -\nu_A \sigma_{22}^{a}$ is tensile positive and opens the crack faces when p_0 is different from zero.

V - CONCLUSION

Non-planar elastic-plastic cracks have been studied. These are of finite extension along x_1 and x_2 and infinite in the x_3- direction, inside an infinitely extended elastic medium, subjected to mixed mode *I+II+III* loading. The loadings, tension σ_{22}^{a} and shears σ_{12}^{a} and σ_{23}^{a}, are applied along the x_2, x_1 and x_3 directions at infinity, respectively. The front of the crack is planar in x_2x_3, has an average elevation $h = h(x_1)$ from Ox_1x_3 and fluctuates weakly there in the form ξ (3). The plastic region is described by $x_2 = h(x_1)$ with a straight front parallel to x_3. The crack and plastic zone are represented by a continuous distribution of three types J of infinitesimal dislocation with Burgers vectors directed along the applied loadings. Distribution functions D_J of straight dislocation arrays corresponding to an elastic-plastic crack π_0 (***Figure* 1*a***), inclined by angle θ_0 with respect to Ox_1x_3 are calculated. Adopting these D_J, explicit expressions of the crack-tip stresses and crack extension force per unit length of the crack front, for the general crack front f (3), are evaluated. Except for a difference in the value of the stress intensity factor, that now depends on the size of the elastic-plastic crack, these quantities agree with those of the isolated crack. Hence both types of crack can be treated in the equal manner.

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