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NON-PLANAR INTERFACE CRACK UNDER GENERAL LOADING III. *DISLOCATION, CRACK-TIP STRESS AND CRACK EXTENSION FORCE*

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ABSTRACT

An analysis is made of cracks at the non-planar interface of bi-elastic materials. We consider large cracks, having travelled macroscopic distances, in large materials. Locally, the crack front ξ can be seen perpendicular to the propagation direction x_1 of the crack. A model is thus analysed in which the non-planar crack fluctuates around an average plane Ox_1x_3 perpendicular to the direction x_2 of the applied tension. Its front $\xi = \xi\,(x_1, x_3)$ spreads in planes x_2x_3 and runs indefinitely in the direction x_3. Its length is finite and goes from $x_1 = -\,a$ to a. We assume general loading, mixed mode $I + II + III$ (tension σ_{22}^a and shears, σ_{12}^a and σ_{23}^a, applied externally at infinity in the directions x_2, x_1 and x_3, respectively). Induced (normal and shear) stresses, which originate from the Poisson effect (acting perpendicularly to the x_2 - direction of the applied tension) in the x_1 and x_3 directions, are considered. The treatment represents the crack by a continuous distribution consisting of three families J ($J = I, II, III$) of dislocations with infinitesimal Burgers vectors $\vec{b}_I = (0, b, 0)$, $\vec{b}_{II} = (b, 0, 0)$ and $\vec{b}_{III} = (0, 0, b)$, respectively. Elastic fields of the dislocations are first given. Then, distribution functions D_J of straight dislocations, corresponding to plane interface cracks, are given. The expressions found do not invoke oscillatory singularities at the tip of the crack; the singularities are those obtained for cracking in a completely homogeneous medium. Adopting these expressions, we could give approximate expressions of crack-tip stresses and crack extension force G for non-planar interface crack with front ξ. Formulas for spatial average $<G>$ of G are provided for special non-planar cracks with segmented and sinusoidal fronts. We compared this extension force G of the crack with that obtained with elastic fields containing oscillatory singularities at the tip of the crack; it is the planar crack which is best documented in mode I loading. A remarkable agreement has been found when neglecting Poisson effect.

Keywords : *linear elasticity, interface sinusoidal dislocations, Galerkin vector, three-dimensional biharmonic functions, singular integral equations, fracture mechanics.*

RÉSUMÉ

Fissure d'interface non plane sous sollicitation extérieure arbitraire III. *Dislocation, contrainte en tête de fissure et force d'extension de fissure*

Une analyse est faite de fissures situées à l'interface non plane de deux solides élastiques. Nous considérons des fissures de grandes tailles, ayant parcouru de larges distances, dans des matériaux de grandes dimensions. Localement, le front de fissure ξ peut être vu perpendiculaire à la direction de propagation x_1 de la fissure. On analyse donc un modèle où la fissure non plane fluctue autour d'un plan moyen Ox_1x_3 perpendiculaire à la direction x_2 de la tension appliquée σ_{22}^a. Son front $\xi = \xi\ (x_1, x_3)$ s'étale dans les plans x_2x_3 et coure indéfiniment dans la direction x_3. Sa longueur est finie et va de $x_1 = -a$ à a. Nous considérons une sollicitation générale, mode mixte *I+ II+ III* (tension σ_{22}^a et cisaillements σ_{12}^a et σ_{23}^a appliqués extérieurement à l'infini dans les directions x_2, x_1 et x_3, respectivement). Sont prises en compte les contraintes induites (normales et de cisaillement) par l'effet Poisson agissant perpendiculairement à la direction x_2 de la tension appliquée et dans les directions x_1 et x_3. Le traitement représente la fissure par une distribution continue constituée de trois familles *J* (*J= I, II, III*) de dislocations de vecteurs de Burgers infinitésimaux $\vec{b}_I = (0, b, 0)$, $\vec{b}_{II} = (b, 0, 0)$ et $\vec{b}_{III} = (0, 0, b)$, respectivement. Des expressions de champs élastiques des dislocations sont d'abord données. Ensuite, on fournit des expressions de fonctions de distribution D_J de dislocations *J* droites correspondant à des fissures d'interface planes. Les expressions trouvées n'invoquent pas de singularités oscillatoires à l'extrémité de la fissure ; les singularités sont celles obtenues pour une fissuration dans un milieu totalement homogène. Nous proposons dans ces conditions des expressions de contraintes en tête de fissure et force d'extension *G* de fissure (par unité de longueur du front de fissure), pour une fissure non plane de front ξ. Des formules pour une moyenne spatiale $<G>$ de *G* sont fournies pour des fissures non planes spéciales dont les fronts sont segmentés et sinusoïdales. Dans le cas d'une fissure d'interface plane sollicitée en tension, la force d'extension de la fissure est en accord remarquable avec celle proposée dans la littérature en négligeant l'effet Poisson.

Mots-clés : *élasticité linéaire, dislocations sinusoïdales d'interface, Vecteur de Galerkin, fonctions biharmoniques à trois dimensions, équation intégrale singulière, mécanique de la rupture.*

I - INTRODUCTION

We consider two elastic solids $R1$ and $R2$ (ν_m and μ_m, m=1 and 2, being their Poisson's ratio and shear modulus, respectively) firmly welded along a non-planar interface R of arbitrary shape. Our goal is to provide crack-tip stress and crack extension force expressions for a crack propagating along R. For large specimens and cracks, we study the following model ***(Figure 1)*** in a Cartesian axis system x_i: the crack along the x_1 - axis (direction of fracture propagation) extends from x_1 = - a to a; its shape in the x_2x_3 planes can be developed in the form of a series of Fourier

$$\xi = \sum_n \left(\xi_n \sin \kappa_n x_3 + \delta_n \cos \kappa_n x_3 \right) \equiv \sum_n A_n \tag{1}$$

where n is a positive integer; ξ_n, δ_n and κ_n are real numbers that depend on position x_1 along the crack length. The media (m = 1 and 2) are assumed to be infinitely large and the crack front extends indefinitely in the x_3 - direction ***(Figure 1)***.

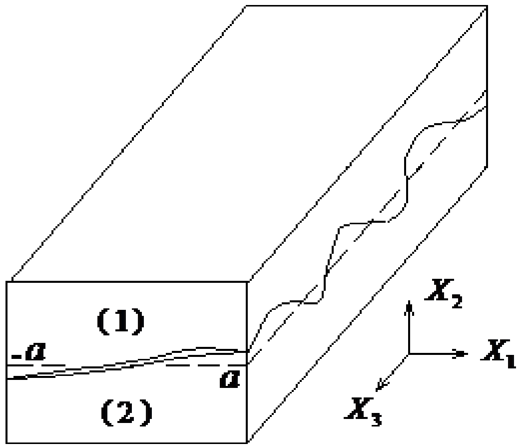

Figure 1 : *Illustration of a crack front in two elastic solids* (1) *and* (2) *welded along a non-planar wavy surface that contains an interface crack. The crack fronts lie in* x_2x_3 *– planes in the form* ξ (1)*; in this geometry, the system is subjected to mixed mode I+II+III loading with the applied tension in the* x_2 *– direction. The average fracture surface (dashed) is shown perpendicular to that direction*

Under arbitrary applied loadings, our method of analysis consists in representing the crack by a continuous distribution of dislocations. The stress field induced by the crack is equivalent to that produced by the dislocations.

The idea that a crack under load is equivalent to a continuous distribution of dislocations of infinitesimal Burgers vectors goes back to Friedel [1, 2], Bilby et al. [3], Bilby and Eshelby [4], to quote only some earlier papers. It is required to determine the dislocation distribution functions at equilibrium under the combined action of the applied stresses and the mutual interaction of the dislocations. From the stress field of a dislocation and by superposition, one obtains the stresses in the surrounding medium produced by the crack and the applied forces. The crack dislocations are virtual entities representing mathematically a crack under load as opposed to the physical dislocations produced by the plastic deformation of the materials as observed by transmission electron microscopy (see for example [5, 6]). It is customary to deal with crack dislocations the Burgers vectors of which are attached to the crack, more precisely, perpendicular (opening of the crack faces) and parallel (sliding and tearing) to the crack plane.

However, another fixed crack dislocation geometry provides the identical expressions for the crack-tip stress and crack extension force [7 - 9]; the Burgers vectors of the dislocations are directed along the applied loading directions whatever the crack plane orientation. In the following, the crack is represented by three dislocation families (*I*, *II* et *III*) whose common shape $\xi(1)$ is perpendicular to x_1 with Burgers vectors $\vec{b}_I = (0, b, 0)$, $\vec{b}_{II} = (b, 0, 0)$ and $\vec{b}_{III} = (0, 0, b)$, parallel to the directions of the applied tension σ_{22}^a and shears σ_{12}^a and σ_{23}^a, respectively. This crack model and method of analysis have been used in the case of an infinitely extended isotropic medium [7 to 14]. The aim of this study is to extend the analysis to non-planar interfacial cracking of bi-materials; interface plane cracks have been the subject of a recent study [15, 16] to which we shall return in Section 5 to discuss the types of singularity (oscillatory and non-oscillatory) of the elastic fields at the tips of cracks, obtained by theoretical analyses.

In parts I [17] and II [18] of this study, elastic field expressions (displacement and stress) of sinusoidal interface dislocations ***(Figure 2)*** perpendicular to x_1, of Burgers vectors $\vec{b}_I$, $\vec{b}_{II}$ and $\vec{b}_{III}$, were given; the fields corresponding to dislocations of form $\xi(1)$ follow from these by superposition. It emerges from these studies that continuity of the elastic fields (displacement and stress) when crossing the interface between media (1) and (2) is not systematic. This is because there are linear relationships between some elastic field components with proportionality coefficients that depend on the elastic constants; when two components are concerned by such relations, the continuity of one at the passage of the interface can cause the discontinuity of the other. In this part III, we shall provide expressions for crack-tip stresses and crack extension force (per unit length of the crack front) for a non-planar crack system ***(Figure 1)***, subjected to general loading, under the requirement that the stresses due to crack dislocations (involved in the crack analysis) be continuous at the crossing of the interface. In what follows, in Section 2, we present the methodology for determining the elastic fields of dislocations and crack analysis. In Section 3 are listed expressions of dislocation elastic fields, distribution functions of interface crack dislocations, crack-tip stresses and crack extension force. Special interface cracks are detailed in Section 4. Sections 5 and 6 are devoted to the discussion and conclusion, respectively.

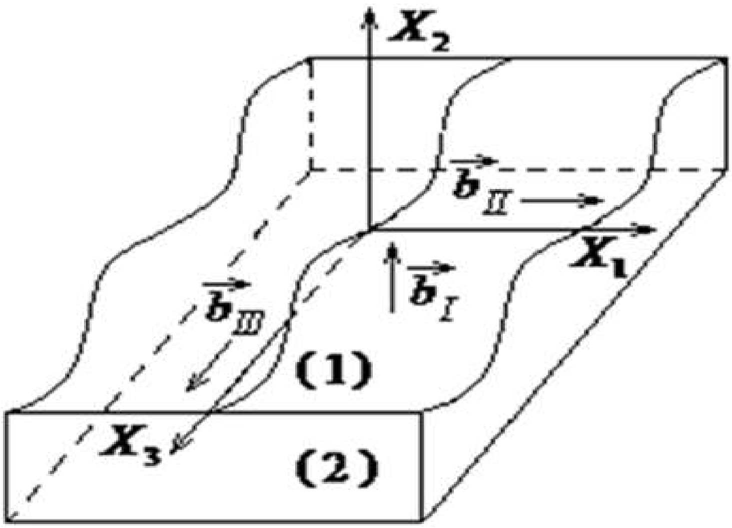

Figure 2 : *Two elastic mediums (1) and (2) welded along a non-planar sinusoidal surface and containing an interface sinusoidal dislocation at the origin. The dislocation lies in the* Ox_2x_3- *plane and runs indefinitely in the* x_3-*direction*

II - METHODOLOGY

II-1. Elastic fields of crack dislocations

The three types *J* (*J*= *I*, *II* and *III*) of crack dislocation considered have equal shape ξ (1) in the Ox_2x_3-plane at the origin, Burgers vectors $\vec{b}_I = (0, b, 0)$, $\vec{b}_{II} = (b, 0, 0)$ and $\vec{b}_{III} = (0, 0, b)$, and run indefinitely in the x_3-direction. Types *I* and *II* are of edge character on average and type *III* screw. The elastic fields (displacement $\vec{u}^{(J)(m)}$ and stress $(\sigma)^{(J)(m)}$) in the bi-material (media m=1 and 2) may be deduced from those (sinusoidal dislocations, ***Figure 2***) with simple form $A_n = \xi_n \sin \kappa_n x_3$ the expressions of which have been described previously [17 to 20]. For sinusoidal dislocations, the elastic fields are (to linear terms in amplitude ξ_n) :

$$\vec{u}^{(J)(m)} = \vec{u}^{(J)(0)(m)} + \vec{u}^{(J)A_n(m)}$$
$$(\sigma)^{(J)(m)} = (\sigma)^{(J)(0)(m)} + (\sigma)^{(J)A_n(m)} .$$

Here $\vec{u}^{(J)(0)(m)}$ and $(\sigma)^{(J)(0)(m)}$ are of zero order; they correspond to the fields of interface straight dislocation (parallel to x_3) at the origin on Ox_1x_3 planar interface with Burgers vector $\vec{b}_J$; $\vec{u}^{(J)A_n(m)}$ and $(\sigma)^{(J)A_n(m)}$ are oscillating parts proportional to A_n or its spatial derivative $\partial A_n / \partial x_3$. When the dislocations exhibit shape ξ (1), the elastic fields take the form

$$\vec{u}^{(J)(m)} = \vec{u}^{(J)(0)(m)} + \vec{u}_{\xi}^{(J)(m)}$$
$$(\sigma)^{(J)(m)} = (\sigma)^{(J)(0)(m)} + (\sigma)_{\xi}^{(J)(m)} ;$$
$$\vec{u}_{\xi}^{(J)(m)} = \sum_n \vec{u}^{(J)A_n(m)}$$
$$(\sigma)_{\xi}^{(J)(m)} = \sum_n (\sigma)^{(J)A_n(m)} . \qquad (2)$$

Here, A_n is that defined in (1). The solution methodology has been detailed [17 to 20]. Because both displacement and stress have similar decompositions, we just write for the stress below; the description is equally valid for the displacement.

$$(\sigma)^{(J)(0)(m)} = (\sigma)^{(J)(0)(m)\infty} - (\sigma)^{(J)(0)(m)W}$$
$$(\sigma)^{(J)A_n(m)} = (\sigma)^{(J)A_n(m)\infty} - (\sigma)^{(J)A_n(m)W} \quad ;$$
$$(\sigma)^{(J)(0)(m)W} = \sum_{l=a \text{ up to } e} \eta_l^{(J)(0)(m)} (\sigma)_l^{(J)(0)(m)V}$$
$$(\sigma)^{(J)A_n(m)W} = \sum_{l=a \text{ up to } e} \eta_l^{(J)A_n(m)} (\sigma)_l^{(J)A_n(m)V} . \qquad (3)$$

The subscript l may take values a up to e (minimum two, a and b; maximum five, a to e) depending on the dislocation J. Expressions with ∞ ($(\sigma)^{(J)(0)(m)\infty} + (\sigma)^{(J)A_n(m)\infty} \equiv (\sigma)^{(J)(m)\infty}$) represent the fields of a sinusoidal dislocation with Burgers vector $\vec{b}_J$ in isotropic infinite medium (m) [7, 13, 21]. Those with W ($(\sigma)^{(J)(0)(m)W} + (\sigma)^{(J)A_n(m)W} \equiv (\sigma)^{(J)(m)W}$) satisfy the equations of equilibrium and are constructed in such a way that the elastic fields of the interface sinusoidal dislocation are continuous at the crossing of the interface and tend to those due to the sinusoidal dislocation in an infinite medium far from the dislocation and interface. $(\sigma)^{(J)(m)W}$ is linear combination of fields with V [17 to 20] (i.e. $(\sigma)_l^{(J)(0)(m)V}$ and $(\sigma)_l^{(J)A_n(m)V}$) that we refer to as "partial elastic fields". They are obtained with the help of Galerkin vectors with three-dimensional biharmonic functions. The proportionality coefficients $\eta_l^{(J)(0)(m)}$ and $\eta_l^{(J)A_n(m)}$ are real numbers that are determined by the requirement that the elastic fields of the interface dislocations (that is to say $(\sigma)^{(J)(0)(m)}$ and $(\sigma)^{(J)A_n(m)}$) are continuous across the interface; we have considered the corresponding equations for spatial positions $P(x_1, x_2= 0, x_3)$ on the average fracture plane only. When these elastic fields are bounded, we have required their linear terms proportional to x_1 to be constant with m= 1 and 2. When the elastic fields contain terms with singularities (of the types Dirac delta $\delta(x_1)$, $1/x_1$, $\ln(x_1)\ldots$), the coefficient of each singularity is set constant with m. In this way, we get many coefficients denoted

$$e_j^{(J)(0)}(m) = e_j^{(J)(0)}(\eta_l^{(J)(0)(m)}, \mu_m, \nu_m)$$
$$e_j^{(J)A_n}(m) = e_j^{(J)A_n}(\eta_l^{(J)A_n(m)}, \mu_m, \nu_m) \quad (4)$$

that depend on $\eta_l^{(J)(0)(m)}$, $\eta_l^{(J)A_n(m)}$ and the elastic constants μ_m shear modulus and ν_m Poisson's ratio. The subscript j may take values from minimum three (j= 1 to 3) up to maximum thirteen (j= 1 to 13). We mention that certain $e_j^{(J)(0)}(m)$ and $e_j^{(J)A_n}(m)$ (in limited numbers) will not be constant with m because they are proportional to others imposed constant with m. In the present study, the components of dislocation stress fields that contribute a non-zero value to the crack extension force or are involved in the crack dislocation distribution functions will be imposed constant with m. These are stresses that contain singularities of the types $\delta(x_1)$ and $1/x_1$. A general approach is to impose on all terms with singularities in the elastic fields (displacement and stress) to be constant with m at the crossing of the interface. There are several ways to find suitable values for $\eta_l^{(J)(0)(m)}$ and $\eta_l^{(J)A_n(m)}$ that make most $e_j^{(J)(0)}(m)$ and $e_j^{(J)A_n}(m)$ constant with m. We can build a system of independent linear equations with as many equations as unknowns $\eta_l^{(J)(0)(m)}$ and $\eta_l^{(J)A_n(m)}$ that can be solved by the classical method with determinant, as performed in [20]. There is another method that allows one to choose the elastic fields one wants imperatively continuous when crossing the interface. First, by direct inspection, linear combinations between the $e_j^{(J)(0)}(m)$ or $e_j^{(J)A_n}(m)$ are sought; then we impose, in progressive manner in number, to some of them to be continuous when crossing the interface. We thus succeed in constructing expressions of $\eta_l^{(J)(0)(m)}$ and $\eta_l^{(J)A_n(m)}$ which ensure an optimum continuity of the elastic fields at the interface (see [18], for example). Frequently, at different stages of this construction procedure, it is required that shear modules and Poisson's ratios of media (1) and (2) be different; hence, the elastic field expressions found in the present study are valid under the conditions $\mu_1 \neq \mu_2$ and $\nu_1 \neq \nu_2$.

II-2. Crack analysis

The crack in ***Figure 1*** is assumed to be filled continuously over the interval $x_1 = -a$ to a with three families J ($J = I$, II and III) of dislocations with shape ξ (1) in x_2x_3- planes and Burgers vectors $\vec{b}_J$ ($\vec{b}_I = (0,b,0)$, $\vec{b}_{II} = (b,0,0)$ and $\vec{b}_{III} = (0,0,b)$). It is understood in our crack analysis that ξ_n, δ_n and κ_n (introduced in (1)) are position dependent along x_1 in the dislocation distributions. The surrounded media (m= 1 and 2) are taken infinite, isotropic, elastic and subjected to uniform remote applied loadings, tension σ_{22}^a and shears σ_{12}^a and σ_{23}^a , at infinity. In addition, we shall consider induced normal and shear stresses originating from the Poisson effect that acts perpendicularly to the tension x_2- direction. The first task is to determine the dislocation distribution functions D_J at equilibrium. D_J (x_1) gives the number of dislocations of family J in a small interval dx_1 about x_1 as $D_J(x_1)\, dx_1$. We generally arrive at a system of singular integral equations involving these functions; when the distribution functions of the dislocations have been found, we can obtain by integration the relative displacement of the faces of the crack, the crack-tip stress and the crack extension force. The surfing point on the interface is P_R (x_1, ξ, x_3). To find the equilibrium dislocation distributions, we may ask for zero total force on the crack faces; this gives

$$\begin{cases} \bar{\sigma}_{12}^{(m)} - \partial\xi/\partial x_1 \bar{\sigma}_{11}^{(m)} - \partial\xi/\partial x_3 \bar{\sigma}_{13}^{(m)} = 0 \\ \bar{\sigma}_{22}^{(m)} - \partial\xi/\partial x_1 \bar{\sigma}_{12}^{(m)} - \partial\xi/\partial x_3 \bar{\sigma}_{23}^{(m)} = 0 \\ \bar{\sigma}_{23}^{(m)} - \partial\xi/\partial x_1 \bar{\sigma}_{13}^{(m)} - \partial\xi/\partial x_3 \bar{\sigma}_{33}^{(m)} = 0 \end{cases} . \qquad (5)$$

$\bar{\sigma}_{ij}^{(m)}$ stands for the total stress at any point (x_1, x_2, x_3) in the surrounding media (m= 1 and 2) and is linked to D_J. In (5), we are concerned with the points of the crack faces only. We write $\bar{\sigma}_{ij}^{(m)}$ as

$$\bar{\sigma}_{ij}^{(m)} = \sigma_{ij}^{A(m)} + \bar{\sigma}_{ij}^{(I)(m)} + \bar{\sigma}_{ij}^{(II)(m)} + \bar{\sigma}_{ij}^{(III)(m)}. \tag{6}$$

We define $\sigma_{ij}^{A(m)}$ and $\bar{\sigma}_{ij}^{(J)(m)}$ successively below. $(\sigma^{A(m)})$ includes the externally applied stresses and induced normal and shear stresses due to the Poisson effect;

$$(\sigma^{A(m)}) = \begin{pmatrix} -\nu_m \sigma_{22}^a & \sigma_{12}^a + \tau_{12}^a & 0 \\ \sigma_{12}^a + \tau_{12}^a & \sigma_{22}^a & \sigma_{23}^a + \tau_{23}^a \\ 0 & \sigma_{23}^a + \tau_{23}^a & -\nu_m \sigma_{22}^a \end{pmatrix}. \tag{7}$$

τ_{12}^a and τ_{23}^a are internal shearing stresses transmitted in the media by the interface as the result of the existence of internal uniform Poisson stresses $(-\nu_1\sigma_{22}^a)$ and $(-\nu_2\sigma_{22}^a)$ in the x_1 and x_3 directions in media (1) and (2), respectively. For spatial position $P_R(x_1, \xi, x_3)$ on the interface, they read [15, 16]

$$\tau_{12}^a = -\bar{\alpha} x_1$$
$$\tau_{23}^a = -\bar{\alpha} x_3;$$
$$\bar{\alpha} = \frac{\nu_s \mu_s}{a_s}\left(\frac{\nu_1}{E_1} - \frac{\nu_2}{E_2}\right)\sigma_{22}^a. \tag{8}$$

The coefficient $\nu_s \mu_s / a_s$ is a characteristic of the interface, E_m (m= 1 and 2) is Young's modulus; we may view μ_s as a shear modulus about the interface, ν_s a constant in the crack analysis (about 0.5) and a_s a slab of material at P_R embracing the interface (measured in the x_2 - direction of the applied tension) that suffers the magnitude of the shearing stress under consideration. $\bar{\sigma}_{ij}^{(J)(m)}$ has the form

$$\bar{\sigma}_{ij}^{(J)(m)}(x_1, x_2, x_3) = \int_{-a}^{a} \sigma_{ij}^{(J)(m)}(x_1 - x_1', x_2, x_3) D_J(x_1') dx_1' ; \tag{9}$$

$\sigma_{ij}^{(J)(m)}$ is the stress field produced by a dislocation of family J located at the origin O. In its general form, (5) requires a numerical resolution. Only simple forms of ξ (special cracks) are tractable. For the planar interface crack, the crack dislocations are straight parallel to x_3; analytical solutions have been obtained recently [15, 16] under modes of applied loading I and I+II based on dislocation stresses taken from [22]. The dislocation distribution functions D_I and D_{II} display an oscillatory singularity at the tip of the crack. When the crack is in an infinitely extended homogeneous medium, we have given approximate solutions D_I and D_{III} under mixed mode $I+III$ loading when $\xi = \xi\ (x_3)$ depends on x_3 only [13]. Analytical D_J values for a planar crack tilted around x_3 by angle θ_0 with respect to Ox_1x_3 in homogeneous medium under mixed mode $I+II+III$ are also reported (see [7, 9] for example). In the present problem of a non-planar interface crack in a bi-material, we can fortunately give approximate expressions of the crack-tip stress and crack extension force with ξ given by (1), taking for D_J, dislocation distributions of planar straight dislocation arrangements (see Section 3 below). ***Figure 3*** is a schematic representation of simple special cracks captured by the modelling. The cracks extend in the x_1- direction, from $x_1 = -a$ to a, and must be considered to run indefinitely in the x_3- direction. The crack shape in planes perpendicular to x_1 is described by ξ (see ***Figure 3 c and d***). The average fracture plane is Ox_1x_3. When $\xi = 0$, the crack dislocations are straight parallel to x_3 and distributed over Ox_1x_3- interface plane.

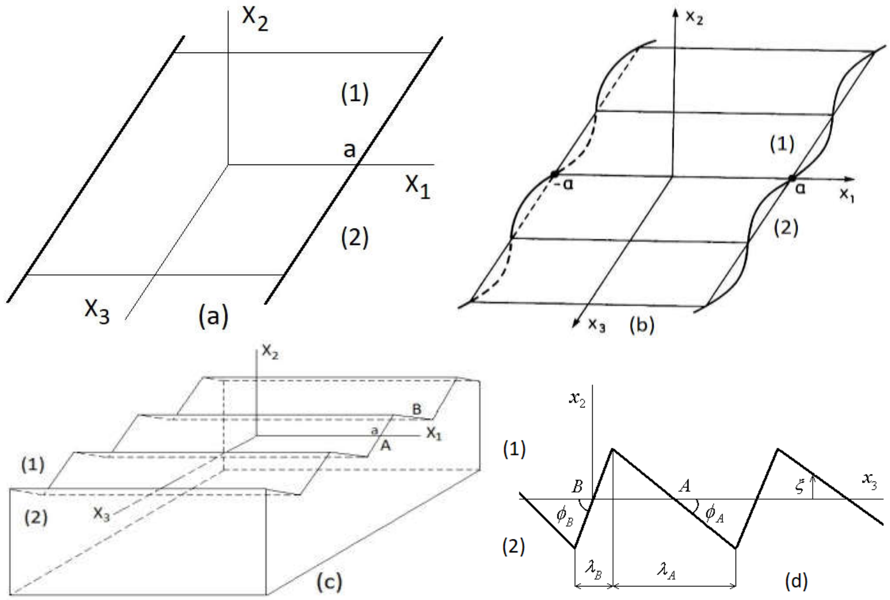

Figure 3 : *Simple special cracks. (a) Planar interface crack with a straight front parallel to x_3 extending from $x_1 = -a$ to a. (b) Wavy crack (fluctuating about Ox_1x_3) in the form of a corrugated sheet with a sinusoidal front independent of x_1. The average plane is illustrated. (c) Non-planar crack fluctuating about Ox_1x_3 and consisting of planar facets; its fronts at $x_1 = \pm a$ lie in x_2x_3- planes. At $x_1 = a$, the crack front is characterized by inclination angles ϕ_A and ϕ_B (see (d)) at positions A and B located on Ox_1x_3. (d) Sketch of the crack front in (c) with B taken as origin. In this geometry (from (a) to (c)), the general loading of the crack systems corresponds to uniform applied σ^a_{22}, σ^a_{12} and σ^a_{23} at infinity in the x_2, x_1 and x_3 directions, respectively*

III - RESULTS

III-1. Displacement and stress fields due to interface dislocation of edge I average character

We consider an interface dislocation with Burgers vector $\vec{b}_I = (0, b, 0)$ lying indefinitely in the x_3- direction and spreading in the x_2x_3- plane at the origin in Fourier series (1). The elastic fields take the form (2). We get at spatial position (x_1, x_2, x_3) ($x_2 \neq 0$, $r^2 = x_1^2 + x_2^2$, $\text{sgn}(x_2) = |x_2| / x_2$):

$$u_1^{(I)(0)(m)} = \frac{C_m(1-2\nu_m)}{2\mu_m}\ln r - \frac{C_m}{2\mu_m r^2}\left(x_1^2 + \text{sgn}(x_2)x_2^2\right) + \frac{x_2^2}{\mu_m r^2}\frac{b\mu_1\mu_2(\mu_1-\mu_2)}{\pi\left[\mu_1^2(3-4\nu_2)-\mu_2^2(3-4\nu_1)\right]},$$

$$u_2^{(I)(0)(m)} = -\frac{b\mu_1\mu_2}{\pi\mu_m\left[\mu_1^2(3-4\nu_2)-\mu_2^2(3-4\nu_1)\right]}\left((\mu_1-\mu_2)\frac{x_1|x_2|}{r^2} + \left(\mu_1[3-2(\nu_m+\nu_2)] - \mu_2[3-2(\nu_m+\nu_1)]\right)\tan^{-1}\frac{x_1}{|x_2|}\right),$$

$$\sigma_{11}^{(I)(0)(m)} = \frac{4b\mu_1\mu_2\,\text{sgn}(x_2)x_1}{\pi\left[\mu_1^2(3-4\nu_2)-\mu_2^2(3-4\nu_1)\right]r^2}\left(\mu_1\nu_2 - \mu_2\nu_1 - (\mu_1-\mu_2)\frac{x_2^2}{r^2}\right),$$

$$\sigma_{22}^{(I)(0)(m)} = \frac{4b\mu_1\mu_2\,\text{sgn}(x_2)x_1}{\pi\left[\mu_1^2(3-4\nu_2)-\mu_2^2(3-4\nu_1)\right]r^2}\left(\mu_1(1-\nu_2) - \mu_2(1-\nu_1) + (\mu_1-\mu_2)\frac{x_2^2}{r^2}\right)$$

$$\sigma_{33}^{(I)(0)(m)} = \frac{4b\nu_m\mu_1\mu_2(\mu_1-\mu_2)\,\text{sgn}(x_2)x_1}{\pi\left[\mu_1^2(3-4\nu_2)-\mu_2^2(3-4\nu_1)\right]r^2},$$

$$\sigma_{12}^{(I)(0)(m)} = \frac{4b\mu_1\mu_2|x_2|}{\pi\left[\mu_1^2(3-4\nu_2)-\mu_2^2(3-4\nu_1)\right]r^2}\left(\mu_1\nu_2 - \mu_2\nu_1 - (\mu_1-\mu_2)\frac{x_2^2}{r^2}\right);$$

for $x_2 = 0$,

$$u_1^{(I)(0)(m)} = -\frac{2b\mu_1\mu_2(\nu_1-\nu_2)}{\pi\left[\mu_1^2(3-4\nu_2)-\mu_2^2(3-4\nu_1)\right]}\ln|x_1|$$

$$u_2^{(I)(0)(m)} = -\text{sgn}(x_1)\frac{b\mu_1\mu_2\left(\mu_1[3-2(\nu_m+\nu_2)] - \mu_2[3-2(\nu_m+\nu_1)]\right)}{2\mu_m\left[\mu_1^2(3-4\nu_2)-\mu_2^2(3-4\nu_1)\right]}$$

$$\sigma_{11}^{(I)(0)(m)} = -\frac{4b\mu_1\mu_2(\nu_1\mu_2 - \nu_2\mu_1)}{\pi\left[\mu_1^2(3-4\nu_2) - \mu_2^2(3-4\nu_1)\right]}\frac{1}{x_1} \equiv -\frac{e_6^{(I)(0)}}{x_1}$$

$$\sigma_{22}^{(I)(0)(m)} = \frac{4b\mu_1\mu_2\left[\mu_1(1-\nu_2) - \mu_2(1-\nu_1)\right]}{\pi\left[\mu_1^2(3-4\nu_2) - \mu_2^2(3-4\nu_1)\right]}\frac{1}{x_1} \equiv \frac{e_3^{(I)(0)}}{x_1}$$

$$\sigma_{33}^{(I)(0)(m)} = \frac{4b\nu_m\mu_1\mu_2\left(\mu_1 - \mu_2\right)}{\pi\left[\mu_1^2(3-4\nu_2) - \mu_2^2(3-4\nu_1)\right]}\frac{1}{x_1} \equiv \frac{e_5^{(I)(0)}(m)}{x_1}$$

$$\sigma_{12}^{(I)(0)(m)} = -\frac{2b\mu_1\mu_2\left[\mu_1(1-2\nu_2) - \mu_2(1-2\nu_1)\right]}{\left[\mu_1^2(3-4\nu_2) - \mu_2^2(3-4\nu_1)\right]}\delta(x_1) \equiv -\eta_c^{(I)(0)}4\pi i Q_c\delta(x_1); \quad (10)$$

$C_m = b\mu_m / 2\pi(1-\nu_m)$, $Q_c = i(\nu_2 C_2 - \nu_1 C_1)/4$, other elastic fields are zero and constant terms are omitted. $\vec{u}^{(I)A_n(m)\infty}$ and $(\sigma)^{(I)A_n(m)\infty}$ are taken from [7, 13].

$$\vec{u}^{(I)A_n(m)W} = \eta_a^{(I)A_n(m)}\vec{u}_a^{(I)A_n(m)V} + \eta_d^{(I)A_n(m)}\vec{u}_d^{(I)A_n(m)V}$$
$$(\sigma)^{(I)A_n(m)W} = \eta_a^{(I)A_n(m)}(\sigma)_a^{(I)A_n(m)V} + \eta_d^{(I)A_n(m)}(\sigma)_d^{(I)A_n(m)V}; \quad (11)$$

$\vec{u}_{a \text{ and } d}^{(I)A_n(m)V}$ and $(\sigma)_{a \text{ and } d}^{(I)A_n(m)V}$ are the partial elastic fields given by [20] (see relations (36) and (39) there);

$$\eta_d^{(I)A_n(2)} = \frac{\text{num}(\eta_d^{(I)A_n(2)})}{\text{denom}(\eta_d^{(I)A_n(2)})},$$
$$\text{num}(\eta_d^{(I)A_n(2)}) = \left[a_{1d}^{(1)} - 2\nu_1\overline{b}^{(1)}\right]\left[C_1\mu_2(1-2\nu_1) - C_2\mu_1(1-2\nu_2)\right]$$
$$+\mu_2\left[a_{1d}^{(1)} + \overline{b}^{(1)}(5-8\nu_1)\right]\left[C_1(1-\nu_1) - C_2(1-\nu_2)\right],$$
$$\text{denom}(\eta_d^{(I)A_n(2)}) = 2i\left\{\mu_1\left[a_{1d}^{(2)} - \overline{b}^{(2)}(5-8\nu_2)\right]\left[a_{1d}^{(1)} - 2\nu_1\overline{b}^{(1)}\right]\right.$$
$$\left.-\mu_2\left[a_{1d}^{(1)} + \overline{b}^{(1)}(5-8\nu_1)\right]\left[a_{1d}^{(2)} + 2\nu_2\overline{b}^{(2)}\right]\right\};$$
$$\eta_d^{(I)A_n(1)} = (\eta_d^{(I)A_n(2)})^{Inv};$$
$$\eta_a^{(I)A_n(2)} = -\eta_d^{(I)A_n(2)}\frac{i\left[\overline{b}^{(2)} + a_{2d}^{(2)}\right]}{a_{1a}^{(2)}},$$
$$\eta_a^{(I)A_n(1)} = (\eta_a^{(I)A_n(2)})^{Inv}. \quad (12)$$

Here, the notation $(...)^{Inv}$ means the value obtained from the expression into the bracket (…) by inverting the elastic constants. The various parameters $a_{1a}^{(m)}$,

$a_{1d}^{(m)}$, $a_{2d}^{(m)}$ and $\overline{b}^{(m)}$ are defined in [20] (see relation (35) there). $\sigma_{23}^{(I)A_n(m)}(x_1,0,x_3)$ is constant with m=1 and 2; it is involved in the crack analysis. Restricting ourselves to terms with $1/x_1$ only, we obtain (to linear terms in A_n or its spatial x_3-derivative)

$$\sigma_{23}^{(I)A_n(m)}(x_1,A_n,x_3)=\frac{\partial A_n}{\partial x_3}\frac{e_9^{(I)A_n}}{x_1},$$

$$e_9^{(I)A_n}=\frac{8i\overline{\overline{b}}\left[\eta_d^{(I)A_n(2)}\mu_1\nu_1(1-\nu_2)+\eta_d^{(I)A_n(1)}\mu_2\nu_2(1-\nu_1)\right]}{\mu_2-\mu_1}; \qquad (13)$$

$$\overline{\overline{b}}=(Q_c-Q_b)/\left[\nu_1(1-\nu_2)+\nu_2(1-\nu_1)\right],\ Q_b=i(C_2-C_1)/4.$$

III-2. Displacement and stress fields due to interface dislocation of edge II average character

The interface dislocation, with Burgers vector $\vec{b}_{II}=(b,0,0)$, lies in the x_3-direction and spreads in the Ox_2x_3- plane in the form of a Fourier series (1). We obtain :

$$u_1^{(II)(0)(m)}=\frac{b}{2\pi}\tan^{-1}\frac{x_2}{x_1}-ie_1^{(II)(0)}\tan^{-1}\frac{x_1}{|x_2|}+\frac{x_1x_2}{r^2}\left(\frac{C_m}{2\mu_m}+(-1)^{m-1}\frac{1}{1-2\nu_m}iV_R^{(II)(0)}\right)$$

,

$$u_2^{(II)(0)(m)}=-\frac{(1-2\nu_m)C_m}{2\mu_m}\ln r+2\ln|x_1|\delta_A(x_2)(-1)^m\left(\frac{C_2-C_1}{4(\mu_2-\mu_1)}+\frac{1-\nu_m}{1-2\nu_m}iV_R^{(II)(0)}\right)$$
$$+\frac{x_2^2}{r^2}\left(\frac{C_m}{2\mu_m}+\frac{1}{1-2\nu_m}iV_R^{(II)(0)}\right),$$

$$\sigma_{11}^{(II)(0)(m)}=-C_m\frac{x_2(x_2^2+3x_1^2)}{r^4}-\left(\frac{x_2}{r^2}+\pi\delta(x_1)\delta_A(x_2)\right)2ie_3^{(II)(0)}(m)$$
$$+\frac{x_2(x_1^2-x_2^2)}{r^4}(-1)^m\frac{\mu_m}{1-2\nu_m}2iV_R^{(II)(0)},$$

$$\sigma_{22}^{(II)(0)(m)}=\frac{x_2(x_1^2-x_2^2)}{r^4}\left[C_m+\frac{(-1)^m\mu_m 2iV_R^{(II)(0)}}{1-2\nu_m}\right]+\left(\frac{x_2}{r^2}+\pi\delta(x_1)\delta_A(x_2)\right)2ie_4^{(II)(0)},$$

$$\sigma_{33}^{(II)(0)(m)}=-2\nu_mC_m\frac{x_2}{r^2}-\left(\frac{x_2}{r^2}+\pi\delta(x_1)\delta_A(x_2)\right)\frac{\nu_m\mu_m}{1-2\nu_m}4iV_R^{(II)(0)},$$

$$\sigma_{12}^{(II)(0)(m)} = \frac{C_m x_1 (x_1^2 - x_2^2)}{r^4} + \frac{x_1(-1)^m}{r^2}\left(\frac{\mu_m (C_2 - C_1)}{\mu_2 - \mu_1} + \frac{x_2^2}{r^2}\frac{\mu_m 4iV_R^{(II)(0)}}{1-2\nu_m}\right), x_2 \neq 0,$$

$$\left(\frac{\Delta V}{V}\right)^{(II)(0)(m)} = \frac{1-2\nu_m}{2(1+\nu_m)\mu_m}\left(\sigma_{11}^{(II)(0)(m)} + \sigma_{22}^{(II)(0)(m)} + \sigma_{33}^{(II)(0)(m)}\right)$$

$$= -\left(\frac{x_2}{r^2} + \pi\delta(x_1)\delta_A(x_2)\right)2iV_R^{(II)(0)} + \frac{x_2(x_1^2 - x_2^2)}{r^4}(-1)^m \frac{2iV_R^{(II)(0)}}{1+\nu_m}$$

$$-\frac{(1-2\nu_m)C_m}{\mu_m}\frac{x_2}{r^2}; \quad (14)$$

$$V_R^{(II)(0)} = \frac{ib(1-2\nu_1)(1-2\nu_2)\left[\mu_2(1-\nu_1) - \mu_1(1-\nu_2)\right]}{4\pi(1-\nu_1)(1-\nu_2)\left[\mu_2(1-2\nu_1) - \mu_1(1-2\nu_2)\right]},$$

$$e_1^{(II)(0)} = \frac{i(C_2 - C_1)}{2(\mu_2 - \mu_1)} + V_R^{(II)(0)},$$

$$e_3^{(II)(0)}(m) = \frac{i\mu_m (C_2 - C_1)}{2(\mu_2 - \mu_1)} + \frac{\mu_m}{1-2\nu_m}V_R^{(II)(0)},$$

$$e_4^{(II)(0)} = \frac{ib\mu_1\mu_2(\nu_2 - \nu_1)\left[\mu_2(1-\nu_1) - \mu_1(1-\nu_2)\right]}{2\pi(1-\nu_1)(1-\nu_2)(\mu_2 - \mu_1)\left[\mu_2(1-2\nu_1) - \mu_1(1-2\nu_2)\right]}. \quad (15)$$

δ_A has the following definition: $\delta_A(x_2) = 0$ when $x_2 \neq 0$ and $\delta_A(x_2) = 1$ when $x_2 = 0$. The only stress component with a singularity $1/x_1$ on the Ox_1x_3- plane is $\sigma_{12}^{(II)(0)(m)}$. Dirac delta function $\delta(x_1)$ appears in $\sigma_{ii}^{(II)(0)(m)}$, i= 1 to 3 and the relative volume variation $(\Delta V/V)^{(II)(0)(m)}$. We have

$$\sigma_{12}^{(II)(0)(m)}(x_1, 0, x_3) = \frac{b\mu_1\mu_2(\nu_1 - \nu_2)}{2\pi(1-\nu_1)(1-\nu_2)(\mu_2 - \mu_1)}\frac{1}{x_1} \equiv \frac{e_6^{(II)(0)}}{x_1}. \quad (16)$$

Expressions (14) correspond to the elastic fields of a straight interface dislocation (parallel to x_3) with a Burgers vector $\vec{b}_{II} = (b,0,0)$. [18] gives similar expressions for the same dislocation. In both cases, the relative variation of volume is non-zero and continuous at the crossing of the interface. What changes is that in [18], $\sigma_{22}^{(II)(0)(m)}$ is discontinuous and $u_2^{(II)(0)(m)}$ continuous across the interface, while in the present work, $\sigma_{22}^{(II)(0)(m)}$ is continuous and $u_2^{(II)(0)(m)}$ discontinuous. In both studies, however, there is an optimum number of continuous elastic fields across the interface. For the oscillating parts of the elastic fields, we have the followings: $\vec{u}^{(II)A_n(m)\infty}$ and $(\sigma)^{(II)A_n(m)\infty}$ are taken from [7];

$$\vec{u}^{(II)A_n(m)W} = \eta_a^{(II)A_n(m)} \vec{u}_a^{(II)A_n(m)V} + \eta_d^{(II)A_n(m)} \vec{u}_d^{(II)A_n(m)V},$$
$$(\sigma)^{(II)A_n(m)W} = \eta_a^{(II)A_n(m)} (\sigma)_a^{(II)A_n(m)V} + \eta_d^{(II)A_n(m)} (\sigma)_d^{(II)A_n(m)V}. \tag{17}$$

$\vec{u}^{(II)A_n(m)W}$ and $(\sigma)^{(II)A_n(m)W}$ are described in [18]; $\vec{u}_{a \text{ and } d}^{(II)A_n(m)V}$ and $(\sigma)_{a \text{ and } d}^{(II)A_n(m)V}$ can be calculated from the Appendix B of this work using relations (B.1) and (B.4), respectively; $\eta_d^{(II)A_n(m)}$ and $\eta_a^{(II)A_n(m)}$ are given by relations (42) and (47) there. We have, to linear terms in the amplitudes of oscillations,

$$\sigma_{13}^{(II)A_n(m)}(x_1, A_n, x_3) = \frac{\partial A_n}{\partial x_3} \frac{e_8^{(II)A_n}}{x_1};$$
$$e_8^{(II)A_n} = \frac{Nu(e_8^{(II)A_n})}{De(e_8^{(II)A_n})},$$
$$Nu(e_8^{(II)A_n}) = 2\mu_1\mu_2\left[(1-2\nu_1)(1-2\nu_2)-(1-\nu_1-\nu_2)^2\right]\left\{2Q_c(1-\nu_1)(1-\nu_2)(C_1-C_2)\right.$$
$$\left. - Q_b\left[\nu_1(1-\nu_2)+\nu_2(1-\nu_1)\right]\left[C_1(1-\nu_1)-C_2(1-\nu_2)\right]\right\},$$
$$De(e_8^{(II)A_n}) = (\nu_1-\nu_2)(\mu_1-\mu_2)\left\{Q_c(1-\nu_1)(1-\nu_2)\left[\mu_1(1-2\nu_2)-\mu_2(1-2\nu_1)\right]\right.$$
$$\left. - Q_b\left[\nu_1(1-\nu_2)+\nu_2(1-\nu_1)\right]\left[\mu_1(1-\nu_1)(1-2\nu_2)-\mu_2(1-\nu_2)(1-2\nu_1)\right]\right\}. \tag{18}$$

Associated $e_7^{(II)A_n}$, appearing below in the expressions of the crack-tip stress and crack extension force, is given in [18] (see relation (46) there).

III-3. Displacement and stress fields due to interface dislocation of screw average character

The interface dislocation has Burgers vector $\vec{b}_{III} = (0, 0, b)$, lies indefinitely in the x_3- direction and spreads in the x_2x_3- plane at the origin in Fourier series (1). We obtain [17] :

$$u_3^{(III)(0)(m)}(\vec{x}) = \frac{b}{2\pi}\tan^{-1}\frac{x_2}{x_1} + \frac{D_1-D_2}{\mu_1+\mu_2}\tan^{-1}\frac{x_1}{|x_2|},$$
$$\sigma_{23}^{(III)(0)(m)}(\vec{x}) = \frac{b\mu_1\mu_2}{\pi(\mu_1+\mu_2)}\frac{x_1}{r^2} \equiv e_3^{(III)(0)}\frac{x_1}{r^2},$$
$$\sigma_{13}^{(III)(0)(m)}(\vec{x}) = -e_3^{(III)(0)}\frac{x_2}{r^2} + \delta_A(x_2)\frac{\mu_m(D_1-D_2)}{\mu_1+\mu_2}\pi\delta(x_1). \tag{19}$$

Associated other elastic field components are zero; $D_m = b\mu_m / 2\pi$.

$\vec{u}^{(III)A_n(m)\infty}$ and $(\sigma)^{(III)A_n(m)\infty}$ are taken from [7, 13].

$$\vec{u}^{(III)A_n(m)W} = \sum_{l=c \text{ to } e} \eta_l^{(III)A_n(m)} \vec{u}_l^{(III)A_n(m)V} ,$$
$$(\sigma)^{(III)A_n(m)W} = \sum_{l=c \text{ to } e} \eta_l^{(III)A_n(m)} (\sigma)_l^{(III)A_n(m)V} ; \tag{20}$$

$\vec{u}^{(III)A_n(m)W}$ and $(\sigma)^{(III)A_n(m)W}$ are described in [17]: $\vec{u}_{c \text{ to } e}^{(III)A_n(m)V}$ and $(\sigma)_{c \text{ to } e}^{(III)A_n(m)V}$ can be calculated from the Appendix B of this work using relations (B.3) to (B.5), respectively; $\eta_{c \text{ to } d}^{(III)A_n(m)}$ correspond to $\eta_{c \text{ to } d}^{A_n(m)}$ in this study where both $\eta_a^{A_n(m)}$ and $\eta_b^{A_n(m)}$ are zero (use relations (38) and (43) in [17]). Compressive stresses $\sigma_{ii}^{(III)A_n(m)}$ (i= 1, 2, 3) have singularities $1/x_1$ when x_2= 0, implying that they are involved in the calculation of the crack extension force below. Associated coefficients $e_j^{(III)A_n}$, constant with m=1 and 2, correspond exactly to $e_j^{A_n}$ in [17] (as given there by (40), for example). We write

$$\sigma_{ii}^{(III)A_n(m)}(x_1, A_n, x_3) = \frac{\partial A_n}{\partial x_3} 4\left(-e_{41}^{(III)A_n}\delta_{i1} + e_{51}^{(III)A_n}\delta_{i2} - e_6^{(III)A_n}\delta_{i3}\right)\frac{1}{x_1} ;$$
$$e_{52}^{(III)A_n} = \eta_c^{(III)A_n(m)} s_{1c}^{(m)}\left[(2-\nu_m)Q_r - \nu_m\right]$$
$$= \frac{1}{3(\Gamma-1)(\nu_2-\nu_1)}\{(\nu_2 C_2 - \nu_1 C_1)[(1-\nu_2)(2-\nu_1) - (1-\nu_1)(2-\nu_2)\Gamma]$$
$$+ (C_2 - C_1)[(1-\nu_2)(1-2\nu_1) - \Gamma(1-\nu_1)(1-2\nu_2)]\},$$
$$e_{51}^{(III)A_n} = -e_{52}^{(III)A_n}/2 ,$$
$$e_{41}^{(III)A_n} = \frac{\nu_1 C_1(1-\nu_1)(1-2\nu_2)\Gamma - \nu_2 C_2(1-\nu_2)(1-2\nu_1)}{2[\Gamma(1-\nu_1)(1-2\nu_2) - (1-\nu_2)(1-2\nu_1)]}$$
$$+ \frac{\nu_1(1-2\nu_2)\Gamma - \nu_2(1-2\nu_1)}{2[\Gamma(1-\nu_1)(1-2\nu_2) - (1-\nu_2)(1-2\nu_1)]} e_{52}^{(III)A_n} ,$$
$$e_6^{(III)A_n} = \frac{\Gamma C_1 - C_2}{2(\Gamma-1)} + \frac{\Gamma(\nu_1 C_1 - \nu_2 C_2)[(1-\nu_1)(2-\nu_2) - (1-\nu_2)(2-\nu_1)]}{2(\Gamma-1)[\Gamma(1-\nu_1)(1-2\nu_2) - (1-\nu_2)(1-2\nu_1)]}$$
$$+ \frac{\nu_1(1-2\nu_2)\Gamma - \nu_2(1-2\nu_1)}{2[\Gamma(1-\nu_1)(1-2\nu_2) - (1-\nu_2)(1-2\nu_1)]} e_{52}^{(III)A_n} . \tag{21}$$

Here δ_{ij} is the Kronecker delta, $\Gamma = \mu_2/\mu_1$, $s_{1c}^{(m)} = (-1)^m iQ_b/2$, $Q_r = Q_c/Q_b$.

III-4. Planar dislocation distributions

It is assumed here that the dislocations are straight parallel to the x_3- direction ($\xi = 0$). We thus have a planar interface crack of finite extension, with straight fronts running indefinitely along x_3 ***(Figure 3a)***. The crack is subjected to mixed mode *I+II+III* with loadings applied at infinity. The condition (5) for the crack faces to be free from tractions becomes (making use of stress expressions from Sections 3.1 to 3.3):

$$\begin{cases} \sigma_{22}^{a} + 2\pi i e_4^{(II)(0)} D_{II}(x_1) + e_3^{(I)(0)} \int_{-a}^{a} \dfrac{D_I(x_1^{'})dx_1^{'}}{x_1 - x_1^{'}} = 0 \\ \sigma_{12}^{a} + \tau_{12}^{a} - \eta_c^{(I)(0)} 4\pi i Q_c D_I(x_1) + e_6^{(II)(0)} \int_{-a}^{a} \dfrac{D_{II}(x_1^{'})dx_1^{'}}{x_1 - x_1^{'}} = 0 \\ \sigma_{23}^{a} + \tau_{23}^{a} + e_3^{(III)(0)} \int_{-a}^{a} \dfrac{D_{III}(x_1^{'})dx_1^{'}}{x_1 - x_1^{'}} = 0 \end{cases} \quad (22)$$

We arrive at a system of three integral equations with Cauchy-type kernels with unknowns D_J. The Cauchy principal values of the integrals will be taken. The solution of the third equation with D_{III} is well known [4] and is given below for zero dislocation content after unloading. Following [15, 16], we can manage the two other equations to obtain

$$\int_{-a}^{a} \left(\sqrt{a^2 - x_1^2} - \beta_A^2 \sqrt{a^2 - x_1^{'2}} \right) \frac{D_{II}(x_1^{'})dx_1^{'}}{x_1^{'} - x_1} = f(x_1);$$

$$f(x_1) = \frac{\sigma_{12}^{a} + \tau_{12}^{a}}{e_6^{(II)(0)}} \sqrt{a^2 - x_1^2} + \frac{i\beta_A^2 \sigma_{22}^{a}}{2e_4^{(II)(0)}} x_1,$$

$$\beta_A^2 = -\frac{e_4^{(II)(0)} \eta_c^{(I)(0)} 8Q_c}{e_3^{(I)(0)} e_6^{(II)(0)}} = 1. \quad (23)$$

We observe that introducing the values for $\eta_c^{(I)(0)} 8Q_c$ and $e_3^{(I)(0)}$ (10) on one hand, $e_4^{(II)(0)}$ (15) and $e_6^{(II)(0)}$ (16) on the other, from the dislocation stress fields (Sections 3.1 and 3.2), we have obtained the surprising value one to β_A^2 independent of the elastic constants. We temporary modify our notation and write (23) as

$$\frac{\sqrt{1-s^2}}{\pi}\int_{-1}^{1}\frac{g(t)dt}{t-s}+\frac{\lambda_A^2}{\pi}\int_{-1}^{1}\frac{\sqrt{1-t^2}\,g(t)dt}{t-s}=\bar{f}(s);$$
$$g(t)=D_{II}(at),$$
$$\bar{f}(s)=\frac{1}{\pi}\left(\frac{(\sigma_{12}^a+\tau_{12}^a)\sqrt{1-s^2}}{e_6^{(II)(0)}}+\frac{i\sigma_{22}^a}{2e_4^{(II)(0)}}s\right),$$
$$\lambda_A=i. \tag{24}$$

The integral equation in (24) takes the form of "Example 43" (page 158) in [23]; by a convincing operational approach, it is shown there that the solution in (24) can be written as

$$g(s)=\frac{\bar{\psi}_1(s)-\bar{\psi}_2(s)}{2\lambda_A\sqrt{1-s^2}} \tag{25}$$

where $\bar{\psi}_1$ and $\bar{\psi}_2$ are solutions of the pair of Cauchy type integral equations

$$\bar{\psi}_n(s)+(\delta_{n1}-\delta_{n2})\frac{\lambda_A}{\pi}\int_{-1}^{1}\frac{\bar{\psi}_n(t)}{t-s}dt=\bar{f}(s)+A\delta_{n1}+B\delta_{n2}\ (n=1\text{ and }2); \tag{26}$$

$g(s)$ satisfies the additional requirement

$$\frac{1}{\pi}\int_{-1}^{1}g(t)\ln\left(\frac{1+t}{1-t}\right)dt=\int_{-1}^{1}\frac{\bar{f}(t)}{\sqrt{1-t^2}}dt; \tag{27}$$

A and *B* are arbitrary constants. Solutions for (26), making use of the Plemelj formula [23, 24], are

$$\bar{\psi}_1(s)=\bar{f}(s)/2$$
$$\bar{\psi}_2(s)=c_0^*\bar{f}(s); \tag{28}$$

c_0^* is an arbitrary constant. From (25), we have $g(s)=c_0^{**}\bar{f}(s)/2i\sqrt{1-s^2}$, and the constant c_0^{**} may be given a value by using (27). We can identify the distribution function $D_{II}(x_1)$ of the dislocation family *II* to the singularity term in *g(s)* because bounded terms will be negligible sufficiently closer to the crack tips. Then, we integrate to obtain the associated D_I value in a similar way as shown in [15, 16]. We gather our results in the following form:

$$D_I(x_1) = \frac{\sigma_{22}^a}{\pi e_3^{(I)(0)}} \frac{x_1}{\sqrt{a^2 - x_1^2}}$$

$$D_{II}(x_1) = \frac{\sigma_{12}^a}{\pi e_6^{(II)(0)*}} \frac{x_1}{\sqrt{a^2 - x_1^2}}$$

$$D_{III}(x_1) \equiv \frac{\sigma_{23}^a + \overline{\tau}_{23}^a}{\pi e_3^{(III)(0)}} \frac{x_1}{\sqrt{a^2 - x_1^2}}; \tag{29}$$

$$\overline{\tau}_{23}^a \equiv \frac{1}{2a}\int_{-a}^{a} \tau_{23}^a dx_3 = 0, \qquad e_6^{(II)(0)*} = e_6^{(II)(0)}\left(1 + 2a\theta_0\overline{\alpha}_0\right),$$

$$\theta_0 = \frac{\mu_2(1-\nu_1) - \mu_1(1-\nu_2)}{\pi\left[\mu_2(1-2\nu_1) - \mu_1(1-2\nu_2)\right]}, \qquad \overline{\alpha}_0 = \frac{\nu_s \mu_s}{a_s}\left(\frac{\nu_1}{E_1} - \frac{\nu_2}{E_2}\right).$$

The consequence of setting $\overline{\tau}_{23}^a = 0$ is that the shearing stress τ_{23}^a plays no role in the theory. The corresponding relative displacements ϕ_J of the faces of the crack in the x_2 ($J=I$), x_1 ($J=II$) and x_3 ($J=III$) directions are

$$\phi_I(x_1) = \frac{\sigma_{22}^a b}{\pi e_3^{(I)(0)}} \sqrt{a^2 - x_1^2}$$

$$\phi_{II}(x_1) = \frac{\sigma_{12}^a b}{\pi e_6^{(II)(0)*}} \sqrt{a^2 - x_1^2}$$

$$\phi_{III}(x_1) = \frac{\sigma_{23}^a b}{\pi e_3^{(III)(0)}} \sqrt{a^2 - x_1^2}. \tag{30}$$

III–5. Stresses about the crack-tip

Ahead of the crack-tip at spatial position $P_R(x_1 = a + s_1, x_2 = \xi, x_3)$, $0 < s_1 << a$, the total stress $\overline{\sigma}_{ij}(P_R)$ (6) is identified to the following formula

$$\overline{\sigma}_{ij}^{(m)}(s_1, x_2, x_3) = \sum_{J=I \text{ to } III} \int_{a-\delta a}^{a} \sigma_{ij}^{(J)(m)}(a + s_1 - x_1^{'}, x_2, x_3) D_J(x_1^{'}) dx_1^{'}, \ \delta a << a. \tag{31}$$

This stress expression means that only those dislocations located about the crack front in x_1 – interval $[a - \delta a\ ,\ a]$ will contribute significantly to the stress at $x_1 = a + s_1$ ahead of the crack tip as s tends to zero; any other contribution will be negligible for a sufficiently small value of s_1. Only dominant terms with singularities in the dislocation stress fields may contribute. Because P_R is outside the crack, stress terms with the Dirac delta function contribute nothing. Only dislocation stress terms with $1/x_1$ contribute and provide well-defined values to the crack extension force. Hence, we restrict ourselves to these singularities only in (31) and take the MacLaurin series expansion of $\sigma_{ij}^{(J)(m)}$ up to terms of first order with respect to x_2 (taken small). Under such conditions, the involved integrals are of the type $\int D_J(x_1^{'})/(a+s_1-x_1^{'})dx_1^{'}$ which is calculated approximately by taking for D_J the straight edge and screw dislocation distributions (29) corresponding to a planar crack. We obtain

$$\bar{\sigma}_{ii}^{(m)}(s_1,x_2,x_3)=\frac{K_I^0}{e_3^{(I)(0)}\sqrt{2\pi s_1}}\left(e_6^{(I)(0)}\delta_{i1}+e_3^{(I)(0)}\delta_{i2}+e_5^{(I)(0)}\delta_{i3}+x_2\frac{\partial^2\xi}{\partial x_3^2}\left(C_m\left[\delta_{i3}-\delta_{i1}\right]\right.\right.$$
$$\left.\left.-\eta_d^{(I)A_n(m)}2i\left\{a_{1d}^{(m)}(\delta_{i3}-\delta_{i2})+(-1)^m\bar{b}^{(m)}\left[3\delta_{i3}-\delta_{i1}+2\nu_m(\delta_{i1}-\delta_{i2}+\delta_{i3})\right]\right\}\right)\right)$$
$$+\frac{\partial\xi}{\partial x_3}\frac{K_{III}^0}{e_3^{(III)(0)}\sqrt{2\pi s_1}}4\left(\delta_{i2}e_{51}^{(III)A_n}-\delta_{i1}e_{41}^{(III)A_n}-\delta_{i3}e_6^{(III)A_n}\right),$$
$$\bar{\sigma}_{12}^{(m)}(s_1,x_2,x_3)=\frac{K_{II}^0}{e_6^{(II)(0)*}\sqrt{2\pi s_1}}\left(e_6^{(I)(0)}-x_2\frac{\partial^2\xi}{\partial x_3^2}\left[C_m+(-1)^m 2(e_7^{(II)A_n}-\eta_d^{(II)A_n(m)}2t_{1d}^{(m)})\right]\right),$$
$$\bar{\sigma}_{13}^{(m)}\left(s_1,x_2,x_3\right)=\frac{\partial\xi}{\partial x_3}\frac{e_8^{(II)A_n}K_{II}^0}{e_6^{(II)(0)*}\sqrt{2\pi s_1}},$$
$$\bar{\sigma}_{23}^{(m)}(s_1,x_2,x_3)=\frac{K_{III}^0}{e_3^{(III)(0)}\sqrt{2\pi s_1}}\left(e_3^{(III)(0)}+x_2\frac{\partial^2\xi}{\partial x_3^2}\left[4e_{41}^{(III)A_n}+\frac{e_{52}^{(III)A_n}4\nu_m(1-Q_r)}{(2-\nu_m)Q_r-\nu_m}\right.\right.$$
$$\left.\left.+\eta_d^{(III)A_n(m)}(-1)^{m-1}\frac{4iQ_b\left[\nu_m r_m Q_r+1-\nu_m\right]}{1-2\nu_m}\right]\right)+\frac{\partial\xi}{\partial x_3}\frac{e_9^{(I)A_n}K_I^0}{e_3^{(I)(0)}\sqrt{2\pi s_1}}. \qquad (32)$$

Again s_1, x_2 and x_3 are arbitrary; $s_1=x_1-a<<a$ $(s_1>0)$ and x_2 is small. $K_I^0=\sigma_{22}^a\sqrt{a\pi}$, $K_{II}^0=\sigma_{12}^a\sqrt{a\pi}$ and $K_{III}^0=\sigma_{23}^a\sqrt{a\pi}$.

III–6. Crack extension force

Our definition of the crack extension force is taken from [4]. A crack of length $2a$ is considered at equilibrium under load. Then, this crack grows almost statically, over a short distance, from one of its ends (say $x_1=a$) while the other end remains fixed. A work associated with a newly created surface element *Δs* is then calculated, which is the product of the elastic forces on that element (just before the displacement of the crack front) by the relative displacement of the faces of the newly created crack through *Δs*. This energy is then divided by *Δs*; it is the limit *G* taken by the ratio of this energy divided by *Δs* when the latter tends to zero which is (by definition) the crack extension force per unit length of the crack front at the point P_0 where *Δs* is located (*Δs* tends to zero while keeping a finite dimension along the crack front is appropriate in the case of rigid crack front displacement). We followed the calculation procedure of *G* [4] in the extension to non-planar cracking [7, 10 to 13, 18].

Consider a planar interface crack in the Ox_1x_3- plane, extending from x_1= -*a* to *a*, with straight fronts parallel to x_3. Under pure modes *J* of applied loading only (*J*= *I, II* and *III*), the crack extension forces $G_{0s}^{J\text{ pure}}$ at $\left(x_1 = a, x_2 = 0, x_3\right)$ read

$$G_{0s}^{I\text{ pure}} = \frac{bK_I^{0^2}}{4\pi e_3^{(I)(0)}},\; G_{0s}^{II\text{ pure}} = \frac{bK_{II}^{0^2}}{4\pi e_6^{(II)(0)}},\; G_{0s}^{III\text{ pure}} = \frac{bK_{III}^{0^2}}{4\pi e_3^{(III)(0)}}. \tag{33}$$

To the same planar crack under general loading (mixed mode *I+II+III*) corresponds the crack extension force G_{0s} given by

$$G_{0s} = G_{0s}^a + G_{0s}^b + G_{0s}^c; \tag{34}$$

$$G_{0s}^a = \frac{bK_I^{0^2}}{4\pi e_3^{(I)(0)}},\quad G_{0s}^b = \frac{bK_{II}^{0'^2}}{4\pi e_6^{(II)(0)}},\quad G_{0s}^c = \frac{bK_{III}^{0^2}}{4\pi e_3^{(III)(0)}},\quad K_{II}^{0'} = K_{II}^0 / \left(1 + 2a\theta_0\bar{\alpha}_0\right).$$

Under general loading of a non-planar crack ***(Figure 1)*** with crack front *ξ* (1), we obtain the reduced crack extension force $\tilde{G}(P_0)$, $P_0 = \left(a, \xi, x_3\right)$, as

$$\tilde{G}(P_0) = G(P_0)/(G_{0s}^a + G_{0s}^b + G_{0s}^c) = \sum_{i,j=1}^{3} \tilde{G}_j^{(i)}(P_0);$$

$$\tilde{G}_1^{(1)} = -\frac{\partial\xi/\partial x_1 M_{12}(\Delta_{123}e_6^{(II)(0)})^{-1}}{\sqrt{1 + (\partial\xi/\partial x_1)^2 + (\partial\xi/\partial x_3)^2}}\left(e_6^{(I)(0)} + \xi\frac{\partial^2\xi}{\partial x_3^2}[-C_m\right.$$

$$\left. + \eta_d^{(I)A_n(m)} 2i(-1)^m(1 - 2\nu_m)\bar{b}^{(m)}\right] - \frac{\partial\xi}{\partial x_3}\frac{4e_{41}^{(III)A_n}e_3^{(I)(0)}}{e_3^{(III)(0)}}M_{13}\right),$$

$$\tilde{G}_2^{(1)} = \frac{M_{12}^2 e_3^{(I)(0)}(\Delta_{123}e_6^{(II)(0)})^{-1}}{\sqrt{1 + (\partial\xi/\partial x_1)^2 + (\partial\xi/\partial x_3)^2}}\left(1 + \frac{\xi}{e_6^{(II)(0)}}\frac{\partial^2\xi}{\partial x_3^2}\right[-C_m$$

$$+(-1)^m 2(\eta_d^{(II)A_n(m)} 2t_{1d}^{(m)} - e_7^{(II)A_n})\Big]\Big]\Big),$$

$$\tilde{G}_3^{(1)} = -\frac{(\partial\xi/\partial x_3)^2 M_{12}^2}{\sqrt{1+(\partial\xi/\partial x_1)^2+(\partial\xi/\partial x_3)^2}} \frac{e_8^{(II)A_n} e_3^{(I)(0)}}{\Delta_{123} e_6^{(II)(0)^2}},$$

$$\tilde{G}_1^{(2)} = -\frac{\partial\xi}{\partial x_1}\frac{e_6^{(II)(0)}}{e_3^{(I)(0)}}\frac{1}{M_{12}}\tilde{G}_2^{(1)},$$

$$\tilde{G}_2^{(2)} = \frac{(\Delta_{123} e_3^{(I)(0)})^{-1}}{\sqrt{1+(\partial\xi/\partial x_1)^2+(\partial\xi/\partial x_3)^2}}\left(e_3^{(I)(0)} + \xi\frac{\partial^2\xi}{\partial x_3^2} 2i\eta_d^{(I)A_n(m)}\right.$$

$$\left.\times\left[a_{1d}^{(m)} + (-1)^m 2\nu_m \bar{b}^{(m)}\right] + \frac{\partial\xi}{\partial x_3}\frac{4e_{51}^{(III)A_n} e_3^{(I)(0)}}{e_3^{(III)(0)}} M_{13}\right),$$

$$\tilde{G}_3^{(2)} = -\frac{\partial\xi/\partial x_3 M_{13}(\Delta_{123} e_3^{(III)(0)})^{-1}}{\sqrt{1+(\partial\xi/\partial x_1)^2+(\partial\xi/\partial x_3)^2}}\left(e_3^{(III)(0)} + \xi\frac{\partial^2\xi}{\partial x_3^2}\left[4e_{41}^{(III)A_n} + \frac{e_{52}^{(III)A_n} 4\nu_m(1-Q_r)}{(2-\nu_m)Q_r - \nu_m}\right.\right.$$

$$\left.\left.+\eta_d^{(III)A_n(m)}(-1)^{m-1}\frac{4iQ_b\left[\nu_m r_m Q_r + 1 - \nu_m\right]}{1-2\nu_m}\right] + \frac{\partial\xi}{\partial x_3}\frac{e_3^{(III)(0)} e_9^{(I)A_n}}{e_3^{(I)(0)} M_{13}}\right),$$

$$\tilde{G}_1^{(3)} = \frac{\partial\xi/\partial x_1}{\partial\xi/\partial x_3}\frac{e_6^{(II)(0)}}{e_3^{(III)(0)}}\frac{M_{13}}{M_{12}}\tilde{G}_3^{(1)},$$

$$\tilde{G}_2^{(3)} = -\frac{M_{13}}{\partial\xi/\partial x_3}\frac{e_3^{(I)(0)}}{e_3^{(III)(0)}}\tilde{G}_3^{(2)},$$

$$\tilde{G}_3^{(3)} = -\frac{\partial\xi/\partial x_3 M_{13}(\Delta_{123} e_3^{(III)(0)})^{-1}}{\sqrt{1+(\partial\xi/\partial x_1)^2+(\partial\xi/\partial x_3)^2}}\left(e_5^{(I)(0)} + \xi\frac{\partial^2\xi}{\partial x_3^2}\left[C_m - \eta_d^{(I)A_n(m)} 2i\right.\right.$$

$$\left.\left.\times\left\{a_{1d}^{(m)} + (-1)^m \bar{b}^{(m)}(3+2\nu_m)\right\}\right] - \frac{\partial\xi}{\partial x_3}\frac{4e_6^{(III)A_n} e_3^{(I)(0)}}{e_3^{(III)(0)}} M_{13}\right),$$

$$\Delta_{123} = 1 + (e_3^{(I)(0)}/e_6^{(II)(0)})M_{12}^2 + (e_3^{(I)(0)}/e_3^{(III)(0)})M_{13}^2. \quad (35)$$

In (35) above, $M_{12} = K_{II}^{0'}/K_I^0$, $M_{13} = K_{III}^0/K_I^0$ and defined is $M_{23} = \sigma_{23}^a/\sigma_{12}^a$. Next, we describe special cracks with numerical applications using epoxy(1)/glass(2) bi-material.

IV - SPECIAL INTERFACE CRACKS

IV-1. Planar crack with a straight front

The simple case corresponds to a planar crack of length $2a$ in the Ox_1x_3- plane, extending from $x_1 = -a$ to a, with a straight front parallel to x_3. When the crack is in a totally homogeneous isotropic medium (m), the crack extension forces $G_0^{J(m)}$ under pure modes J of applied loading only ($J = I, II$ and III) read

$$G_0^{J(m)} = \frac{b}{4\pi C_m}\left(\delta_{JI} K_I^{0^2} + \delta_{JII} K_{II}^{0^2} + \delta_{JIII} K_{III}^{0^2} / (1-\nu_m)\right). \tag{36}$$

Hence, with $G_{0s}^{J\text{ pure}}$ (33), we have

$$G_{0s}^{J\text{ pure}} / G_0^{J(m)} = C_m \left(\delta_{JI} \frac{1}{e_3^{(I)(0)}} + \delta_{JII} \frac{1}{e_6^{(II)(0)}} + \delta_{JIII} \frac{1-\nu_m}{e_3^{(III)(0)}} \right). \tag{37}$$

For numerical application throughout, we take media epoxy/glass as system (1)/(2). The parameters taken from [25] are: $E_1 = 2.03$ GPa, $\nu_1 = 0.37$; $E_2 = 68.95$ GPa, $\nu_2 = 0.20$; $E_i = 2\mu_i(1+\nu_i)$, $a/a_s = 10$, $\mu_s = (\mu_1 + \mu_2)/2$, $\nu_s = 0.5$ [15]. We have

$$G_{0s}^{J\text{ pure}} / G_0^{J(1)} = 0.5\delta_{JI} + 4.6\delta_{JII} + 0.5\delta_{JIII} \tag{38}$$

$$G_{0s}^{J\text{ pure}} / G_0^{J(2)} = 15.1\delta_{JI} + 140\delta_{JII} + 20\delta_{JIII}. \tag{39}$$

Under such conditions, further extension of the crack initially located on the interface would be in the epoxy (outside the interface), except for pure mode *II*. Again, with the same interface planar crack (as defined above) but now under general loading (mixed mode *I+II+III*), the crack extension force G_{0s} is given by (34). Defining the normalized crack extension force $\tilde{G}_{0s}^{(m)}$ as

$$\tilde{G}_{0s}^{(m)} = G_{0s} / \left(G_0^{I(m)} + G_0^{II(m)} + G_0^{III(m)}\right),$$

we have

$$\tilde{G}_{0s}^{(m)} = \frac{C_m\left[1 + (e_3^{(I)(0)} / e_6^{(II)(0)}) M_{12}^2 + (e_3^{(I)(0)} / e_3^{(III)(0)}) M_{13}^2\right]}{e_3^{(I)(0)}\left[1 + m_{12}^2 + M_{13}^2 / (1-\nu_m)\right]}; \tag{40}$$

$m_{12} = \sigma_{12}^a / \sigma_{22}^a = K_{II}^0 / K_I^0$, $M_{12} = K_{II}^{0'} / K_I^0 = m_{12} / (1 + 2a\theta_0 \bar{\alpha}_0)$.

$\tilde{G}_{0s}^{(m)}$ is displayed in ***Figure 4*** for epoxy(1) / glass(2) bi-material. Pure mode I loading is achieved when both m_{12} and M_{13} are equal to zero. Pure modes II and III are obtained when m_{12} and M_{13} tend to infinity, respectively. It is concluded that the value of the crack extension force is smaller along the interface as compared to its values in the homogeneous medium (1).

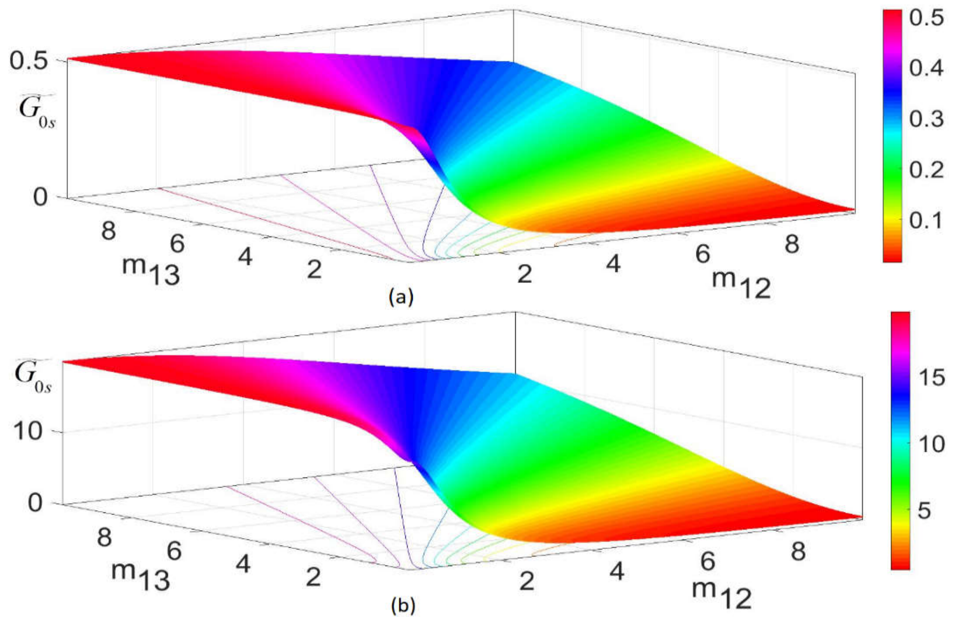

Figure 4 : *Normalized crack extension force* $\tilde{G}_{0s}^{(m)}$ *(40) (planar crack under mixed mode I+II+III loading including Poisson effects) as a function of applied shear/tension ratios* $m_{12} = \sigma_{12}^a / \sigma_{22}^a$ *and* $m_{13} = \sigma_{23}^a / \sigma_{22}^a = M_{13}$ *for (a) m=1 epoxy and (b) m=2 glass (see text)*

IV–2. Non-planar crack with a segmented front

The considered crack has a front that consists of straight segments. It fluctuates around the Ox_1x_3- plane and $\xi = \xi(x_3)$ is independent of x_1 ***(Figure 3c and d)***. We describe ξ below taking locally B as origin as in ***Figure 3d***. ξ is then odd and $(2\lambda = \lambda_A + \lambda_B)$ – periodical with respect to x_3 where λ_A and λ_B ***(Figure 3d)*** are the projected length along x_3 of planar facets at A and B respectively. ξ is given by :

$$\xi = \tan\phi_B x_3 , \qquad |x_3| \le \lambda_B/2$$
$$= \tan\phi_A(-x_3+\lambda) , \qquad x_3 \in [\lambda_B/2,\ \lambda_B/2+\lambda_A] . \tag{41}$$

We assume general loading (mixed mode *I*+*II*+*III*), and next consider successively the reduced crack extension force $\tilde{G}(P_0)$ (35), now denoted $\tilde{G}_{sv}$, express the spatial average $<\tilde{G}_{sv}>$ defined as $<\tilde{G}_{sv}> = (1/2\lambda)\int_0^{2\lambda}\tilde{G}_{sv}dx_3$, and ultimately the condition for an extremum of $<\tilde{G}_{sv}>$. $\tilde{G}_{sv}$ on a period reads: for $|x_3| < \lambda_B/2$,

$$\tilde{G}_{sv} = \frac{\cos\phi_B}{\Delta_{123}}\left(1+\frac{e_3^{(I)(0)}}{e_6^{(II)(0)}}M_{12}^2+\frac{e_3^{(I)(0)}}{e_3^{(III)(0)}}M_{13}^2+\left[-e_3^{(III)(0)}+4e_{51}^{(III)A_n}+e_9^{(I)A_n}\right.\right.$$
$$\left.\left.-e_5^{(I)(0)}\right]\frac{M_{13}}{e_3^{(III)(0)}}\tan\phi_B-\left[\frac{e_9^{(I)A_n}}{e_3^{(I)(0)^2}}+\frac{e_8^{(II)A_n}}{e_6^{(II)(0)^2}}M_{12}^2-\frac{4e_6^{(III)A_n}}{e_3^{(III)(0)^2}}M_{13}^2\right]e_3^{(I)(0)}\tan^2\phi_B\right);$$

for $x_3 \in \left]\lambda_B/2, \lambda_B/2+\lambda_A\right]$,

$$\tilde{G}_{sv} = \frac{\cos\phi_A}{\Delta_{123}}\left(1+\frac{e_3^{(I)(0)}}{e_6^{(II)(0)}}M_{12}^2+\frac{e_3^{(I)(0)}}{e_3^{(III)(0)}}M_{13}^2-\left[-e_3^{(III)(0)}+4e_{51}^{(III)A_n}+e_9^{(I)A_n}\right.\right.$$
$$\left.\left.-e_5^{(I)(0)}\right]\frac{M_{13}}{e_3^{(III)(0)}}\tan\phi_A-\left[\frac{e_9^{(I)A_n}}{e_3^{(I)(0)^2}}+\frac{e_8^{(II)A_n}}{e_6^{(II)(0)^2}}M_{12}^2-\frac{4e_6^{(III)A_n}}{e_3^{(III)(0)^2}}M_{13}^2\right]e_3^{(I)(0)}\tan^2\phi_A\right). \tag{42}$$

For given M_{12} , M_{13} and crack profile (ϕ_A, ϕ_B), (42) provides the reduced crack extension force at an arbitrary spatial position $P_0(a, \xi, x_3)$ on the segmented crack front. $\tilde{G}_{sv}$ takes constant values on facets at *A* and *B* and does not depend on media numeration *m*, except for a value ν_m in $e_5^{(I)(0)}$ (10); hence, it is almost unchanged by inverting the elastic constants. The average $<\tilde{G}_{sv}>$ may be written as

$$<\tilde{G}_{sv}> = \frac{1}{\Delta_{123}}\left(\Delta_{123}v_0-\left[e_3^{(III)(0)}-4e_{51}^{(III)A_n}-e_9^{(I)A_n}\right.\right.$$
$$\left.\left.+e_5^{(I)(0)}\right]\frac{M_{13}}{e_3^{(III)(0)}}v_1-\left[\frac{e_9^{(I)A_n}}{e_3^{(I)(0)^2}}+\frac{e_8^{(II)A_n}}{e_6^{(II)(0)^2}}M_{12}^2-\frac{4e_6^{(III)A_n}}{e_3^{(III)(0)^2}}M_{13}^2\right]e_3^{(I)(0)}v_2\right); \tag{43}$$

$$v_0 = \left(1/(p_A + p_B)\right)\left(p_A/\sqrt{1+p_B^2} + p_B/\sqrt{1+p_A^2}\right),$$
$$v_1 = \left(p_A p_B/(p_A + p_B)\right)\left(1/\sqrt{1+p_B^2} - 1/\sqrt{1+p_A^2}\right),$$
$$v_2 = \left(p_A p_B/(p_A + p_B)\right)\left(p_A/\sqrt{1+p_A^2} + p_B/\sqrt{1+p_B^2}\right).$$

Here $p_A = \tan\phi_A$, $p_B = \tan\phi_B$. We restrict ourselves to the condition for an extremum for $< \tilde{G}_{sv} >$ with respect to ϕ_A by cancelling $\partial < \tilde{G}_{sv} > /\partial\phi_A$. We have

$$\partial < \tilde{G}_{sv} > /\partial\phi_A = \frac{\partial p_A/\partial\phi_A}{\Delta_{123}}\Bigg(\Delta_{123} v_0^{'} - \Big[e_3^{(III)(0)} - 4e_{51}^{(III)A_n} - e_9^{(I)A_n}$$
$$+ e_5^{(I)(0)}\Big]\frac{M_{13}}{e_3^{(III)(0)}} v_1^{'} - \left[\frac{e_9^{(I)A_n}}{e_3^{(I)(0)^2}} + \frac{e_8^{(II)A_n}}{e_6^{(II)(0)^2}} M_{12}^2 - \frac{4e_6^{(III)A_n}}{e_3^{(III)(0)^2}} M_{13}^2\right] e_3^{(I)(0)} v_2^{'}\Bigg); \qquad (44)$$
$$v_0^{'} = \left(p_B/(p_A + p_B)\right)\left((p_A + p_B)^{-1}\left[1/\sqrt{1+p_B^2} - 1/\sqrt{1+p_A^2}\right] - p_A/(1+p_A^2)^{3/2}\right),$$
$$v_1^{'} = \left(p_B/(p_A + p_B)\right)\left(p_B(p_A + p_B)^{-1}\left[1/\sqrt{1+p_B^2} - 1/\sqrt{1+p_A^2}\right] + p_A^2/(1+p_A^2)^{3/2}\right),$$
$$v_2^{'} = \left(p_B/(p_A + p_B)\right)\left(p_B(p_A + p_B)^{-1}\left[p_A/\sqrt{1+p_A^2} + p_B/\sqrt{1+p_B^2}\right] + p_A/(1+p_A^2)^{3/2}\right)$$

$\partial < \tilde{G}_{sv} > /\partial\phi_A = 0$ leads to finding the roots of a polynomial of order 2 in M_{13} ; this gives

$$M_{13} = (-b^* + \sqrt{\Delta^*})/2a^*; \qquad (45)$$
$$\Delta^* = b^{*2} - 4a^* c^*,$$
$$a^* = (e_3^{(I)(0)} e_6^{(II)(0)})^2 \left(e_3^{(III)(0)} v_0^{'} + 4e_6^{(III)A_n} v_2^{'}\right),$$
$$b^* = e_3^{(I)(0)} e_3^{(III)(0)} e_6^{(II)(0)^2} \left(-e_3^{(III)(0)} - e_5^{(I)(0)} + 4e_{51}^{(III)A_n} + e_9^{(I)A_n}\right) v_1^{'},$$
$$c^* = e_3^{(I)(0)} e_6^{(II)(0)} e_3^{(III)(0)^2} \left(e_6^{(II)(0)} + e_3^{(I)(0)} M_{12}^2\right) v_0^{'}$$
$$- e_3^{(III)(0)^2} \left(e_6^{(II)(0)^2} e_9^{(I)A_n} + e_3^{(I)(0)^2} e_8^{(II)A_n} M_{12}^2\right) v_2^{'}.$$

The numerical application is for the bi-material, epoxy (1) / glass (2). We recall that $a/a_s = 10$, $M_{13} = \sigma_{23}^a/\sigma_{22}^a$, $M_{12} = m_{12}/22$ and $m_{12} = \sigma_{12}^a/\sigma_{22}^a$. For M_{12} lower than certain value (not determined but, say 0.5 approximately), $< \tilde{G}_{sv} >$

(43) increases with ϕ_A and ϕ_B from values close to one to values as large as 60, ***Figure 5***. No extremum is present. For larger M_{12}, maximums are present at relatively small M_{13}, ***Figure 6 a and b***. Negative values correspond to forbidden motion of the crack. The behaviour at larger M_{13} is presented on ***Figure 6c***; extremum as given by (45) could be minimum there. Values (M_{13}, ϕ_A) (45) at fixed ϕ_B and M_{12}, corresponding to extremums on the curves $<\tilde{G}_{sv}>(\phi_A)$ are displayed on ***Figure 7***.

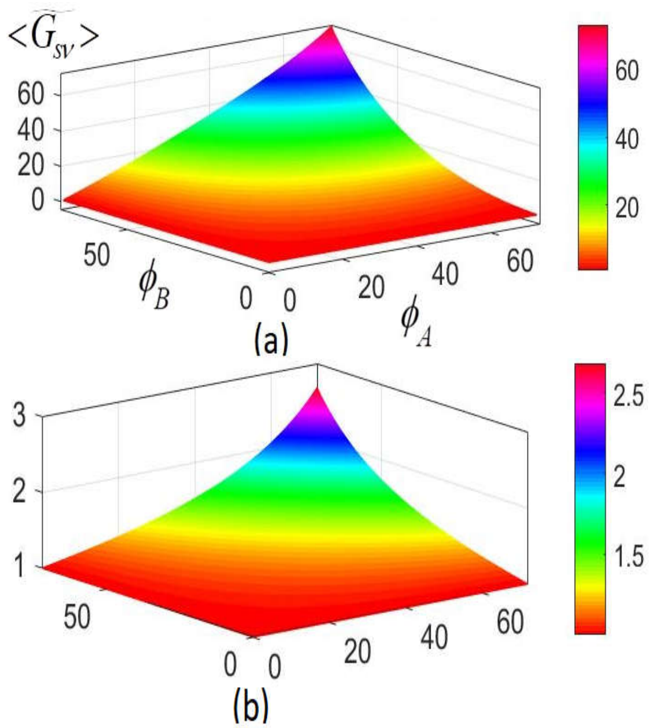

Figure 5 : *Surfaces* $<\tilde{G}_{sv}>(\phi_A,\phi_B)$ (43) *with associated contours for a non-planar interface crack with segmented fronts in bi-material epoxy / glass: (a)* $M_{12} = 0.001$ *and* $M_{13} = 1$*; (b)* $M_{12} = 0.5$ *and* $M_{13} = 0.001$. *ϕ_A and ϕ_B are in degrees*

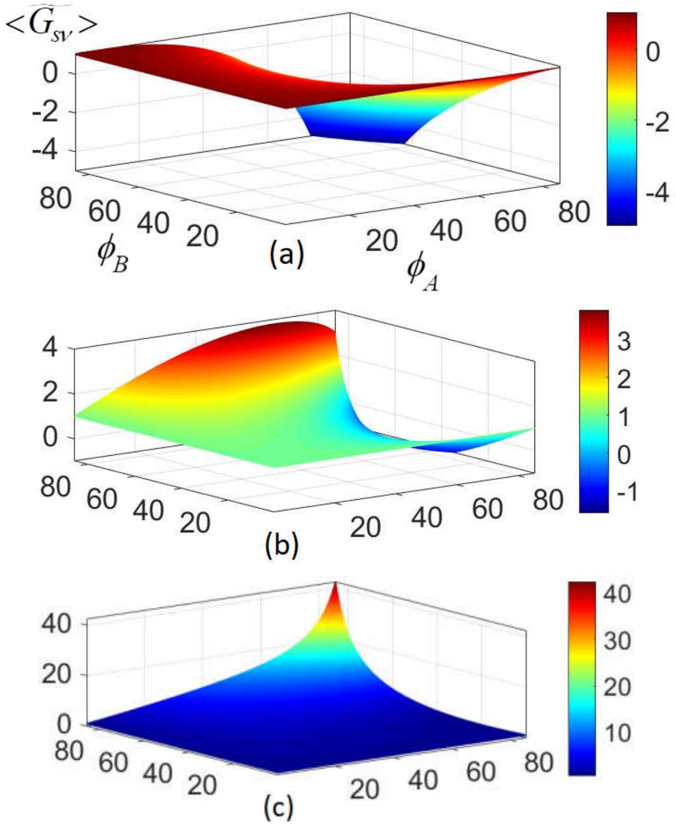

Figure 6 : *Surfaces* $<\tilde{G}_{sv}>(\phi_A,\phi_B)$ (43) *for a non-planar interface crack with segmented fronts in bi-material epoxy / glass with* $M_{12}=1$*: (a)* $M_{13}=0.3$*; (b)* $M_{13}=0.6$ *and (c)* $M_{13}=1$. ϕ_A *and* ϕ_B *are in degrees*

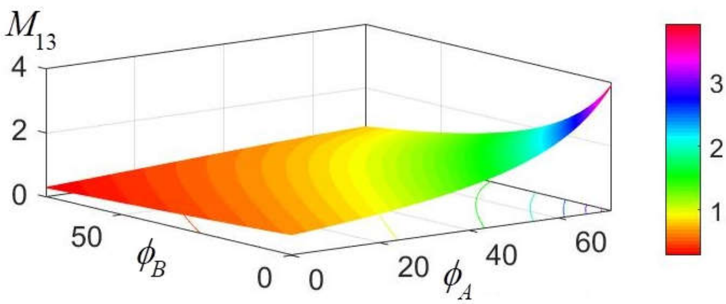

Figure 7 : *Surface* $M_{13}\,(\phi_A,\phi_B)$ (45) *with associated contours for a non-planar interface crack with segmented fronts in bi-material epoxy / glass with* $M_{12}=1$. ϕ_A *and* ϕ_B *are in degrees*

IV–3. Non-planar crack with a sinusoidal front

Here, the crack has a sinusoidal front $\xi = \xi(x_3)$ in x_2x_3- plane that is independent of x_1 ***(Figure 3b)***, in the form

$\xi / \xi_c = \sin \kappa_c x_3 = \sin 2\pi \bar{x}_3$

where $\bar{x}_3 = x_3 / \lambda_c$ and $\lambda_c = 2\pi / \kappa_c$ is the wavelength. In the following, we express the reduced crack extension force (35), denoted $\tilde{G}_{sc}$ for the sinusoidal crack, the spatial average $< \tilde{G}_{sc} >= (1/\lambda_c)\int_0^{\lambda_c} \tilde{G}_{sc} dx_3$ and finally conditions under which $< \tilde{G}_{sc} >$ has extremum. $\tilde{G}_{sc}$ depends on the product $\xi_c \kappa_c = \tan \phi_c$; ϕ_c, the crack-front inclination angle, is the acute angle measured in the plane perpendicular to the crack propagation direction between the crack front and the average fracture plane. We obtain

$$\tilde{G}_{sc} = \frac{1}{\sqrt{1 + \tan^2 \phi_c \cos^2 2\pi \bar{x}_3}} \Big(1 + A_{s1} \tan^2 \phi_c + 2A_{s2}^{(m)} \tan^2 \phi_c \sin^2 2\pi \bar{x}_3 + A_{s3}^{(m)} \tan \phi_c \cos 2\pi \bar{x}_3 + A_{s4}^{(m)} \tan^3 \phi_c \cos 2\pi \bar{x}_3 \sin^2 2\pi \bar{x}_3 \Big). \quad (46)$$

We have:

$$A_{s1} = -\frac{e_9^{(I)A_n} e_3^{(III)(0)^2} e_6^{(II)(0)^2} + e_8^{(II)A_n} e_3^{(I)(0)^2} e_3^{(III)(0)^2} M_{12}^2 - 4e_6^{(III)A_n} e_3^{(I)(0)^2} e_6^{(II)(0)^2} M_{13}^2}{e_3^{(I)(0)} e_3^{(III)(0)^2} e_6^{(II)(0)^2} \Delta_{123}};$$

$$A_{s2}^{(m)} = \frac{num(A_{s2}^{(m)})}{2e_3^{(I)(0)} e_3^{(III)(0)^2} e_6^{(II)(0)^2} \Delta_{123}},$$

$$\begin{aligned} num(A_{s2}^{(m)}) = {} & e_6^{(II)(0)^2} e_3^{(III)(0)^2} \left\{ e_9^{(I)A_n} - \eta_d^{(I)A_n(m)} 2i \left[a_{1d}^{(m)} + (-1)^m 2\nu_m \bar{b}^{(m)} \right] \right\} \\ & + M_{12}^2 e_3^{(I)(0)^2} e_3^{(III)(0)^2} \left\{ e_8^{(II)A_n} + (-1)^m 2e_7^{(II)A_n} + 2\mu_m V_{R1}^{(II)A_n} / (1 - 2\nu_m) \right\} \\ & - M_{13}^2 e_3^{(I)(0)^2} e_6^{(II)(0)^2} \left(4e_6^{(III)A_n} + 4e_{41}^{(III)A_n} + \frac{e_{52}^{(III)A_n} 4\nu_m (1 - Q_r)}{(2 - \nu_m) Q_r - \nu_m} \right. \\ & \left. + (-1)^{m-1} \eta_d^{(III)A_n(m)} \frac{4iQ_b (\nu_m r_m Q_r + 1 - \nu_m)}{1 - 2\nu_m} \right); \end{aligned}$$

$$A_{s3}^{(m)} = \frac{M_{13}}{e_3^{(III)(0)} \Delta_{123}} \left(e_9^{(I)A_n} - e_5^{(I)(0)} + 4e_{51}^{(III)A_n} - e_3^{(III)(0)} \right);$$

$$A_{s4}^{(m)} = \frac{M_{13}}{e_3^{(III)(0)} \Delta_{123}} \left(4e_{41}^{(III)A_n} + \frac{(-1)^{m-1} \eta_d^{(III)A_n(m)} 4iQ_b (\nu_m r_m Q_r + 1 - \nu_m)}{1 - 2\nu_m} \right.$$

$$+\frac{e_{52}^{(III)A_n} 4\nu_m(1-Q_r)}{(2-\nu_m)Q_r-\nu_m}+C_m-\eta_d^{(I)A_n(m)} 2i\left[a_{1d}^{(m)}+(-1)^m\bar{b}^{(m)}(3+2\nu_m)\right]\Bigg). \quad (47)$$

For given M_{12}, M_{13} and crack profile ϕ_c, (46) gives the reduced crack extension force at arbitrary spatial position $P_0(a,\xi_c \sin\kappa_c x_3, x_3)$ on the crack front. We observe that A_{s1} is unchanged by inverting the elastic constants, as also is $A_{s3}^{(m)}$ except in the latter case for a factor ν_m in $e_5^{(I)(0)}$ (10). $A_{s2}^{(m)}$ and $A_{s4}^{(m)}$, in the forms we arrived at, seem not unchanged by inverting the elastic constants. We observe that by inverting the elastic constants in $A_{s2}^{(1)}$ and $A_{s4}^{(1)}$, we obtain $A_{s2}^{(2)}$ and $A_{s4}^{(2)}$, respectively. Hence, one can work with m=1 for half-space (1) in the crack extension force expression (46) and invert the elastic constants to obtain the results applicable for half-space (2). One expects both expressions to have similar magnitudes. In the calculation of $<\tilde{G}_{sc}>$, terms with $\tan\phi_c$ and $\tan^3\phi_c$ (odd powers) in $\tilde{G}_{sc}$ (46) contribute nothing. We get

$$<\tilde{G}_{sc}>=\cos\phi_c\Big(\left[1+A_{s1}\tan^2\phi_c\right]F(1/2,1/2;1;\sin^2\phi_c) + A_{s2}^{(m)}\tan^2\phi_c F(3/2,1/2;2;\sin^2\phi_c)\Big) \quad (48)$$

where F is Gauss's hypergeometric function. The condition for extremum of $<\tilde{G}_{sc}>(\phi_c)$ is $\partial<\tilde{G}_{sc}>/\partial\phi_c=0$. We have

$$\partial<\tilde{G}_{sc}>/\partial\phi_c=\cos\phi_c\left(u_{s0}+u_{s1}A_{s1}+u_{s2}A_{s2}^{(m)}\right); \quad (49)$$

$$u_{s0}=-\tan\phi_c F(1/2,1/2;1;\sin^2\phi_c)+\sin 2\phi_c F(3/2,3/2;2;\sin^2\phi_c)/4,$$

$$u_{s1}=\tan\phi_c\Big[(2+\tan^2\phi_c)F(1/2,1/2;1;\sin^2\phi_c) + \sin 2\phi_c \tan\phi_c F(3/2,3/2;2;\sin^2\phi_c)/4\Big],$$

$$u_{s2}=\tan\phi_c\Big[(2+\tan^2\phi_c)F(3/2,1/2;2;\sin^2\phi_c) + 3\sin 2\phi_c \tan\phi_c F(5/2,3/2;3;\sin^2\phi_c)/8\Big].$$

Given M_{12}, $\partial<\tilde{G}_{sc}>/\partial\phi_c=0$ corresponds to finding the roots of a polynomial of order 2 in M_{13}; this gives

$$M_{13}^2 = -\frac{1}{num_{13}}\left(num_0 + M_{12}^2 num_{12}\right); \tag{50}$$

$$num_{13} = e_3^{(I)(0)^2} e_6^{(II)(0)^2}\left(u_{s0}2e_3^{(III)(0)} + u_{s1}8e_6^{(III)A_n} - u_{s2}\left[\frac{e_{52}^{(III)A_n}4\nu_m(1-Q_r)}{(2-\nu_m)Q_r-\nu_m}\right.\right.$$

$$\left.\left. +4e_6^{(III)A_n} + 4e_{41}^{(III)A_n} + \frac{(-1)^{m-1}\eta_d^{(III)A_n(m)}4iQ_b(\nu_m r_m Q_r + 1 - \nu_m)}{1-2\nu_m}\right]\right),$$

$$num_0 = e_3^{(III)(0)^2} e_6^{(II)(0)^2}\left(u_{s0}2e_3^{(I)(0)} - u_{s1}2e_9^{(I)A_n}\right.$$

$$\left. +u_{s2}\left\{e_9^{(I)A_n} - \eta_d^{(I)A_n(m)}2i\left[a_{1d}^{(m)} + (-1)^m 2\nu_m \bar{b}^{(m)}\right]\right\}\right),$$

$$num_{12} = e_3^{(I)(0)^2} e_3^{(III)(0)^2}\left(u_{s0}2e_6^{(II)(0)} - u_{s1}2e_8^{(II)A_n}\right.$$

$$\left. +u_{s2}\left\{e_8^{(II)A_n} + (-1)^m 2e_7^{(II)A_n} + 2\mu_m V_{R1}^{(II)A_n}/(1-2\nu_m)\right\}\right).$$

Given M_{12} and M_{13}, $<\tilde{G}_{sc}>$ (48) is a function of ϕ_c that may contain extremums, the associated ϕ_c values of which are given by (50). ***Figures 8 and 9*** are plots of $<\tilde{G}_{sc}>$ (48) and pairs (ϕ_c, M_{13}) (50) (given M_{12}) respectively. Gauss's hypergeometric functions are restricted to the nine first terms of their respective series. On ***Figure 8***, for each value of M_{12}, two behaviours of $<\tilde{G}_{sc}>$ are clearly distinguished: at the lowest values of M_{13}, $<\tilde{G}_{sc}>$ increases with ϕ_c, from the value 1 (ϕ_c=0) to values that can exceed 100 ($\phi_c \approx 80^\circ$); at the highest values of M_{13}, $<\tilde{G}_{sc}>$ decreases with ϕ_c, from 1 to negative values. Cracks that produce negative values cannot propagate. At intermediate values of M_{13}, extremums of $<\tilde{G}_{sc}>$ are expected, presumably minimums. These extremums are given in ***Figure 9***.

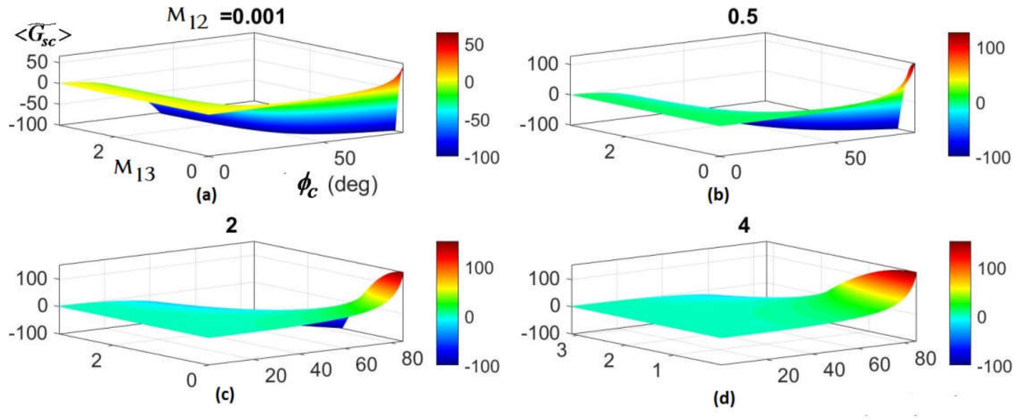

Figure 8 : $<\tilde{G}_{sc}>$ *(48)(m= 1), for a non planar crack with sinusoidal fronts, as a function of* (ϕ_c, M_{13}) *in bi-material epoxy(1) / glass(2) with four different values of* M_{12} *: (a)* $M_{12} = 0.001$*, (b) 0.5, (c) 2 and (d) 4*

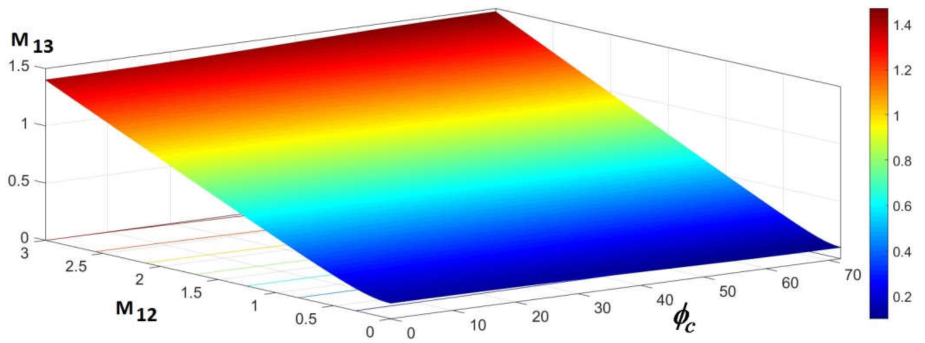

Figure 9 : M_{13} *(50) (m= 1) as a function of* (ϕ_c, M_{12}) *for non-planar crack with sinusoidal fronts in bi- material epoxy / glass*

V - DISCUSSION

The expressions for the crack dislocation distributions D_J (29), crack tip stresses (32) and crack extension force (35) are similar in forms to those for a crack in a totally homogeneous medium. We stress that when $\beta_A^2 = 1$ (23), no oscillatory solution of the elastic fields at the tip of the interface crack exists; this can be checked directly by introducing a function of the form $\overline{\psi}_n(s) = c_0^* \left(1+s / \; 1-s\right)^{i\delta} \; \overline{f}(s)$ in (26). We can write β_A^2 as

$$\beta_A^2 = -\frac{1}{(\pi x_1 \delta(x_1))^2} \frac{\sigma_{12}^{(I)(0)(m)} \sigma_{22}^{(II)(0)(m)}}{\sigma_{12}^{(II)(0)(m)} \sigma_{22}^{(I)(0)(m)}} = 1, \tag{51}$$

independent of media (1) and (2). In the relation (51) above, the dislocation stresses are measured on the interface plane at $P(x_1, x_2 = 0, x_3)$ and identified to their singular terms proportional to $1/x_1$ and $\delta(x_1)$. In recent studies [15, 16], we have analysed a planar interface crack under mode *I* and mixed mode *I+II* loadings (considering shearing stresses on the interface originating from Poisson effect) and represented the crack by continuous distributions of straight edge dislocations. The dislocation stress fields on the interface (positions $P(x_1, x_2 = 0, x_3)$) were taken from [22]:

- Edges with $\vec{b}_{II} = (b, 0, 0)$ (Family *II*)

$$\sigma_{12}^{(II)}(x_1, 0, x_3) = \frac{bC}{\pi}\frac{1}{x_1},$$
$$\sigma_{22}^{(II)}(x_1, 0, x_3) = -b\beta C\delta(x_1);$$

- Edges with $\vec{b}_I = (0, b, 0)$ (Family *I*)

$$\sigma_{12}^{(I)}(x_1, 0, x_3) = b\beta C\delta(x_1),$$
$$\sigma_{22}^{(I)}(x_1, 0, x_3) = \frac{bC}{\pi}\frac{1}{x_1}. \tag{52}$$

Moreover,

$$C = \frac{2\mu_1(1-\alpha)}{(\kappa_1+1)(1-\beta^2)} = \frac{2\mu_2(1+\alpha)}{(\kappa_2+1)(1-\beta^2)}$$

with

$$\alpha = \frac{\mu_1(\kappa_2+1) - \mu_2(\kappa_1+1)}{\mu_1(\kappa_2+1) + \mu_2(\kappa_1+1)}, \quad -1 \le \alpha \le 1$$

$$\beta = \frac{\mu_1(\kappa_2-1) - \mu_2(\kappa_1-1)}{\mu_1(\kappa_2+1) + \mu_2(\kappa_1+1)}, \quad -\frac{1}{2} \le \beta \le \frac{1}{2} \tag{53}$$

where $\kappa_i = 3 - 4\nu_i$. We arrived at a singular integral equation identical to that in (23), but with β^2 in place of β_A^2. The corresponding dislocation distributions, D_I and D_{II}, and the stresses carry oscillatory singularities at the tip of the cracks. In addition, individual crack extension forces, associated with modes *I* and *II*, are ill defined, oscillating indefinitely in their limiting values (see (34) and (31) in [15, 16]). However, the total energy release rate *G* is well defined and reads in mixed mode *I+II*:

$$G = \frac{a\pi(1+4\delta^2)}{4C}\left((\sigma_{22}^a - a\delta\bar{\alpha})^2 + \left(\sigma_{12}^a - \frac{a\bar{\alpha}}{2}\right)^2\right); \qquad (54)$$

$$\delta = \frac{1}{2\pi}\ln\left(\frac{1+\beta}{1-\beta}\right).$$

The oscillatory crack-tip elastic fields and crack extension force agree with previous studies in absence of Poisson effect [26 to 30]. For certain combinations of materials, $\beta = 0$: under such conditions, the oscillatory character of the elastic fields at the tip of the interface cracks disappears and we recover a behaviour like that described in the present study. We observe (52) that the same spatial dependence is given to $\sigma_{22}^{(I)}$ and $\sigma_{12}^{(II)}$ on the interface. We are facing two different interface straight edges *I* and *II*. The ways these independent dislocations are introduced into the bi-material are very different: for the interface climb edge *I*, one can displace one part of material (1) only (say $x_1>0$, $x_2>0$) with respect to ($x_1<0$, $x_2>0$) by *b* whilst for the interface glide edge *II*, the displacement may be performed along the interface (say, $x_1<0$, $x_2>0$) with respect to ($x_1<0$, $x_2<0$) by *b*. Now, taking each dislocation separately in the bi-material, why their stresses must have the identical spatial dependencies on the interface, *i.e*, σ_{12} (interface glide edge *II*) = σ_{22} (interface climb edge *I*)? This behavior is observed when both dislocations are in an infinitely extended isotropic medium. For the bi-material, this result is not obvious. In a similar way (52), except for a minus sign, why σ_{22} (interface glide edge *II*) = - σ_{12} (interface climb edge *I*)? These are so regardless of the types of material involve in the bi-material. This contrasts with the results of the present study where $\sigma_{22}^{(I)(0)}(x_1,0,x_3) = e_3^{(I)(0)}/x_1$ (10) and $\sigma_{12}^{(II)(0)}(x_1,0,x_3) = e_6^{(II)(0)}/x_1$ (16) are different; a similar remark applies with $\sigma_{12}^{(I)(0)}$ (10) and $\sigma_{22}^{(II)(0)}$ (16). Both studies, [22, 15, 16] and the present one, use similar spatial dependences (i.e. $1/x_1$ and $\delta(x_1)$) for the stresses but with different coefficients. Here the coefficients $e_3^{(I)(0)}$, $\eta_c^{(I)(0)}4\pi i Q_c$ (10), $e_6^{(II)(0)}$ (16) and $e_4^{(II)(0)}$ (15) are unchanged by inverting the elastic constants; this ensures continuity of the stresses at the crossing of the interface. In contrast, β(53) changes sign on inverting the elastic constants in stress coefficients (52) suggesting that associated stresses are discontinuous on the interface. Hence, the presence of Dirac delta function $\delta(x_1)$ on $x_2 = 0$ in the interface dislocation stress fields and imposed continuity conditions when crossing the interface provide the result $\beta_A^2 = 1$.

We assume the planar crack to be submitted to mixed mode *I+II* loading only, and defined $\hat{G}_{0s} = G_{0s} / G$ where G_{0s} and G are given by (34) and (54), respectively. This gives

$$\hat{G}_{0s} = \frac{bC}{\pi e_3^{(I)(0)}\left(1+4\delta^2\right)}\left(\frac{1+M_{12}^2(e_3^{(I)(0)} / e_6^{(II)(0)})}{(1-a\delta\bar{\alpha}_0)^2+(m_{12}-a\bar{\alpha}_0/2)^2}\right); \tag{55}$$

$m_{12} = \sigma_{12}^a / \sigma_{22}^a$, $M_{12} = m_{12} / (1+2a\theta_0\bar{\alpha}_0)$. ***Figure 10*** is a plot of $\hat{G}_{0s}(m_{12})$ (55). $\hat{G}_{0s}$ is smaller than 1.

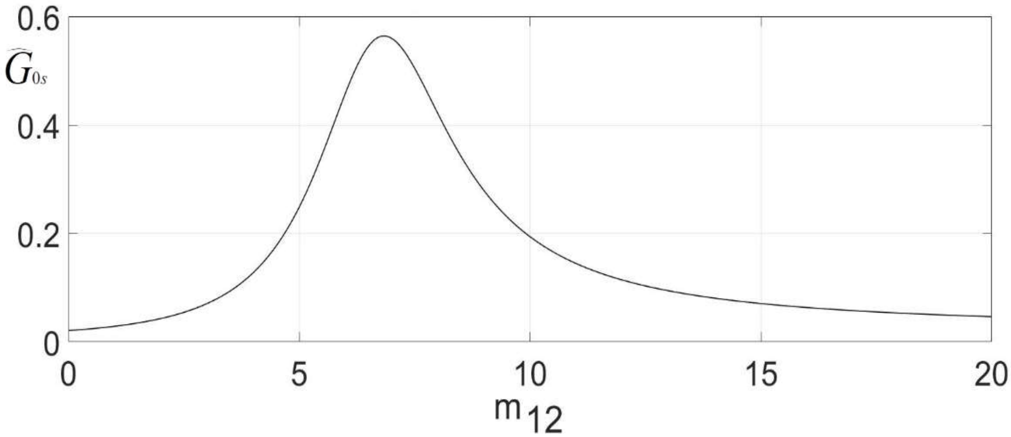

Figure 10 : *$\hat{G}_{0s}$ (55) as a function of $m_{12} = \sigma_{12}^a / \sigma_{22}^a$; mixed mode I+ II loading of planar interface crack in bi-material epoxy (1) / glass (2)*

Figure 11 displays $\hat{G}_{0s}$(55), neglecting Poisson shearing stresses in x_1 direction. Remarkably, $\hat{G}_{0s} \cong 0.98$ under mode *I* loading only ($m_{12} = 0$) suggesting that both, G_{0s} and G, predict the same behaviour of the interface planar crack in epoxy/glass under mode *I* loading, without Poisson effect. In fact, from (55) and $\hat{G}_{0s} \cong 0.98$, the coefficient $bC / (\pi e_3^{(I)(0)}(1+4\delta^2)) \cong 0.98$ is close to one, implying that

$$e_3^{(I)(0)} \approx \frac{bC}{\pi\left(1+4\delta^2\right)} . \tag{56}$$

Other material combination alumina (1) / nickel (2) confirms (56). This indicates that (34) (i.e. G_{0s}^{a}) and (54) are nearly identical under pure mode *I* loading, in absence of Poisson effect. Because for the planar crack, the mode that opens the crack faces is what matters for crack extension, the present study and those that involve oscillatory singularities of elastic fields at the tip of the interface crack, predict the same behaviour of planar interface cracks in real materials. The theory (38) predicts that further extension of a crack initially located on the planar interface of epoxy/ glass would be in epoxy; this is confirmed by the observation [31]. Now, under pure mode *II* loading of planar interface crack in absence of Poisson effect, (55) gives

$$\hat{G}_{0s} = \frac{bC}{\pi e_6^{(II)(0)}\left(1+4\delta^2\right)} (\textit{Pure mode II loading}). \tag{57}$$

This cannot be close to one because, as seen from ***Figure 11*** for m_{12} large, $e_3^{(I)(0)}$ and $e_6^{(II)(0)}$ have different magnitudes.

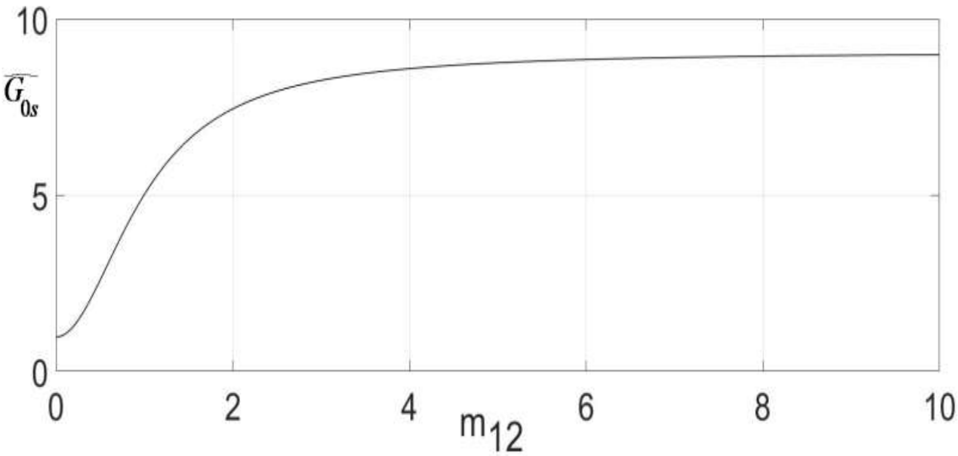

Figure 11 : $\hat{G}_{0s}$ *(55) as a function of* $m_{12} = \sigma_{12}^{a} / \sigma_{22}^{a}$*, in absence of Poisson shearing stresses along x1 on the interface,* $M_{12} = m_{12}$ *; mixed mode I+ II loading of planar interface crack in bi-material epoxy (1) / glass (2)*

VI - CONCLUSION

We have studied a model of cracking of a non-plane interface of a bi-elastic solid material. The crack is macroscopic in size with an oscillatory form front ξ (small in magnitude) located in a plane perpendicular to the x_1 direction of fracture propagation. We considered general loading (mixed mode $I + II + III$) and induced normal and shearing stresses in the bi-material as the results of the contractions due to the Poisson effect in the mediums (1) and (2) acting perpendicularly to the x_2 direction of the applied tension. The shearing stress τ_{23}^{a} plays no role in the theory, on average. We represented the crack by a continuous distribution made of 3 families J of dislocation having the form ξ. Expressions of the elastic fields (displacement and stress) have been given in the linear approximation with respect to the oscillation amplitudes associated with ξ with the concern that the dislocation stresses involved in the dislocation distribution functions D_J at the equilibrium and the crack extension force are continuous at the crossing of the interface. The analytical expressions of the D_J obtained are those of planar arrangements of straight dislocations corresponding to a plane interface crack. They do not contain oscillatory singularities at the end of the crack; it is rather the type of singularity (29) observed in cracking in a homogeneous medium. Adopting these expressions, we could give approximate expressions of crack-tip stresses and crack extension force for non-planar interface crack with front ξ ($x_1 = a$, x_3). We compared this extension force of the crack with that obtained with elastic fields containing oscillatory singularities at the tip of the crack; it is the planar crack which is best documented in mode I loading. A remarkable agreement has been found.

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