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NON-PLANAR CRACK UNDER GENERAL LOADING AND INDUCED NORMAL STRESSES DUE TO POISSON EFFECT

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ABSTRACT

In the present study, we consider a large non-planar crack model corresponding to a cracking over large distances and of arbitrary shape in an elastic isotropic medium. The system is solicited in mixed mode I+II+III. We include in the analysis the induced normal stresses, which result from the Poisson effect, acting perpendicularly to the direction of applied tension. The method of analysis consists in representing the non-planar crack by a continuous distribution of infinitesimal dislocations. Basic dislocations involved in the treatment are sinusoidal edge and screw dislocations the Burgers vectors of which are aligned along the directions of the modes I, II and III of applied loadings. Expressions for the stresses at the crack-tip and the crack extension force *G* (per unit length of the crack front) are given. The study shows that the Poisson effect increases significantly *G*. This data can be considered for a better comparison with experiment.

Keywords : *elasticity, fracture mechanics, dislocation, crack-tip stress, energy release rate.*

RÉSUMÉ

Fissure non plane sous sollicitation extérieure arbitraire et en présence des contraintes normales induites par l'effet Poisson

Dans la présente étude, nous considérons une fissure non plane de grande taille, correspondant à une fissuration sur de grandes distances, et de forme arbitraire dans un milieu isotrope élastique. Le système est sollicité en mode mixte I+II+III. Nous incluons dans l'analyse, des contraintes normales induites qui résultent de la contraction, par effet Poisson, agissant perpendiculairement à la direction de la tension appliquée. La méthode d'analyse consiste à représenter la fissure non-plane par une distribution continue de dislocations infinitésimales. Les dislocations de base impliquées dans le traitement sont des dislocations sinusoïdales coins et vis dont les vecteurs de Burgers sont alignés suivant les directions des modes de sollicitation I, II et III. Nous proposons dans ces conditions, des expressions pour les contraintes en tête de fissure et la force d'extension de la fissure. Il ressort de l'étude que l'effet Poisson augmente nettement la force d'extension de la fissure. Cette donnée est à prendre en considération pour une meilleure confrontation avec l'expérience.

Mots-clés : *élasticité, mécanique de la rupture, dislocation, force d'extension de la fissure, contrainte en tête de fissure.*

I - INTRODUCTION

We propose to analyse the propagation conditions of a non-planar crack of any shape under arbitrary external applied loading (mixed mode I + II + III) by including internal normal stresses which result from the contraction due to the Poisson effect acting perpendicularly to the applied tension direction. The Poisson effect is common to most materials under load, which means that the mechanical parts in use in the various human activities are the seat of normal induced stresses. As an illustration, consider a homogeneous rectangular fracture specimen with large dimensions to which is attached a Cartesian coordinate system x_i ***(Figure 1)***. Initially, the specimen is dashed. Under the action of uniform applied tension σ_{22}^{a}, the specimen extends in the x_2 – direction but shrinks according to the Poisson effect in the x_1 and x_3 directions. The shape then taken by the specimen is in solid ***(Figure 1)***.

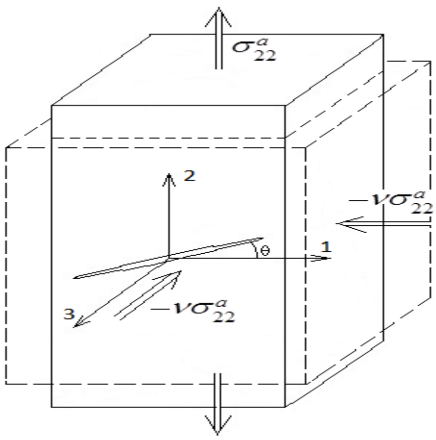

Figure 1 : *Fracture specimen loaded in tension σ_{22}^{a} along x_2. The induced normal stresses along the directions x_1 and x_3 with magnitude $\nu\sigma_{22}^{a}$ (originating from the Poisson effect) are illustrated*

Inside the material, there are normal induced stresses $\sigma_{11}^{a} = -\nu\sigma_{22}^{a}$ and $\sigma_{33}^{a} = -\nu\sigma_{22}^{a}$ in the x_1 and x_3 directions, ν being the Poisson's ratio. We consider a planar crack, the centre of which will be taken as origin *O*. The crack is of finite extension along x_1 and infinite with a straight front on x_3 ; it is inclined around x_3 by angle θ with respect to Ox_1x_3. When $\theta = 0$, the crack is in Ox_1x_3 and parallel to the compression $\sigma_{11}^{a} = -\nu\sigma_{22}^{a}$ direction. Further extension of the crack corresponds to a relative displacement of the faces of the crack along x_2 only; under such conditions, the force due to σ_{11}^{a} does not work and consequently contributes zero to the crack extension force. However, when $\theta \neq 0$, σ_{11}^{a} contributes effectively to the crack extension force with virtually a non-negligible value because $\sigma_{11}^{a}/\sigma_{22}^{a} = \nu$ in magnitude (about 33% in isotropic material). A similar comment can be done with $\sigma_{33}^{a} = -\nu\sigma_{22}^{a}$ when the planar crack is inclined around the x_1 direction by angle θ . Let us add to the above constraints uniform applied shear stresses σ_{12}^{a} and σ_{23}^{a}, parallel to x_1x_3 and in directions x_1 and x_3, respectively. The fracture sample is then externally solicited in mixed mode I+II+III (general loading).

Under such conditions, the broken surface is non-planar and the associated crack exhibits a non-straight oscillating front when viewed in planes perpendicular to the local crack propagation direction (see [1 - 3] and references therein). The fractured surface involves planar portions that are tilted around axes x_1 and x_3, implying that normal induced stresses contribute non-zero values to the crack extension force during the fracture process. The aim of the present study is to provide expressions for the crack-tip stresses and the crack extension force corresponding to non-planar cracks under general loading. A previous work that neglects Poisson effect is available [3]. The present one includes the normal induced stresses that originate from the Poisson effect associated with the tension loading. The interest of such a study is to have a more general expression of the crack extension force *G*. Crack configurations for which *G* is maximum, are those that are compared with experimental findings. We stress that articles dealing with cracks which include normal stresses due to Poisson effect are uncommon in the literature. In Section 2, we present the methodology of the analysis. Sections 3 and 4 are devoted to results, and discussion and conclusion, respectively.

II - METHODOLOGY

The non-planar crack model and associated analysis correspond to those of [3]. The crack is of finite extension along x_1 ($|x_1| \leq a$) and infinite along x_3. Its shape *f* in the planes x_2x_3 is developed in Fourier series form

$$f = \sum_n (\xi_n \sin \kappa_n x_3 + \delta_n \cos \kappa_n x_3) + h(x_1) \equiv \xi(x_1, x_3) + h(x_1)\,. \tag{1}$$

Here *n* is a positive integer; *h*, ξ_n, δ_n and κ_n are real that are x_1 – dependent in the crack analysis. As in all modelling where the crack front runs indefinitely, this crack model is devoted to fracture propagation over large distances. We would like to stress that because the crack model is non-planar from the beginning, the present description appears more general than a modelling that starts with a planar starter crack. This is because the arbitrarily chosen planar starter crack must move first over macroscopic distance, changing gradually its shape from planar to non-planar, in order to achieve crack configurations that effectively correspond to the externally applied loadings. We assume an infinite isotropic elastic medium, subjected to uniform applied tension σ_{22}^a and shears σ_{12}^a and σ_{23}^a , at infinity. In addition, the treatment includes normal induced stresses $\sigma_{11}^a = -\nu\sigma_{22}^a$ and $\sigma_{33}^a = -\nu\sigma_{22}^a$ originating from Poisson effect. Our method of analysis [1 to 10] consists in representing the crack by a continuous distribution of infinitesimal dislocation families. In the present work, the dislocations have form *f* (1). We consider three dislocation families in the distribution [3]. Families $i = 1$ and 2 are edges (I and II) on average with Burgers vectors $\vec{b}_I = (0, b, 0)$ and $\vec{b}_{II} = (b, 0, 0)$ respectively, and family 3 consists of screws with $\vec{b}_{III} = (0, 0, b)$. The three families respond essentially to modes of loading I, II and III, respectively. The dislocation distribution function $D_i(x_1)$ gives the number of dislocations of family i in a small interval dx_1 about x_1 as $D_i(x_1)dx_1$. To find the dislocation distributions, we ask the crack faces to be traction-free; this gives

$$\begin{cases} \overline{\sigma}_{12} - \partial f / \partial x_1 \overline{\sigma}_{11} - \partial f / \partial x_3 \overline{\sigma}_{13} = 0 \\ \overline{\sigma}_{22} - \partial f / \partial x_1 \overline{\sigma}_{12} - \partial f / \partial x_3 \overline{\sigma}_{23} = 0 \\ \overline{\sigma}_{23} - \partial f / \partial x_1 \overline{\sigma}_{13} - \partial f / \partial x_3 \overline{\sigma}_{33} = 0 \end{cases}. \tag{2}$$

$\overline{\sigma}_{ij}$ stands for the total stress at any point (x_1, x_2, x_3) in the medium and is linked to D_i. In (2), we are concerned with the points of the crack faces only. We write $\overline{\sigma}_{ij}$ as

$$\overline{\sigma}_{ij} = \sigma_{ij}^a + \overline{\sigma}_{ij}^{(1)} + \overline{\sigma}_{ij}^{(2)} + \overline{\sigma}_{ij}^{(3)}\,; \tag{3}$$

(σ^a) is the externally applied stress tensor including normal induced stresses from Poisson effect,

$$(\sigma^a) = \begin{pmatrix} -\nu\sigma^a_{22} & \sigma^a_{12} & 0 \\ \sigma^a_{12} & \sigma^a_{22} & \sigma^a_{23} \\ 0 & \sigma^a_{23} & -\nu\sigma^a_{22} \end{pmatrix},$$

$$\overline{\sigma}^{(n)}_{ij}(x_1, x_2, x_3) = \int_{-a}^{a} \sigma^{(n)}_{ij}(x_1 - x_1', x_2, x_3) D_n(x_1') dx_1' \quad (n = 1, 2 \text{ and } 3). \tag{4}$$

$\sigma^{(n)}_{ij}$ (n = 1, 2 and 3) is the stress field produced by a dislocation displaced by ($x_2 = h$) from the origin with Burgers vector $(0, b, 0)$, $(b, 0, 0)$ or $(0, 0, b)$. $\sigma^{(n)}_{ij}$ may be taken from [3]. When the dislocation distributions $D_i(x_1)$ ($i = 1$ to 3) are known, the relative displacement of the faces of the crack, the crack-tip stresses and the crack extension force are derived by integrations (see [2, 3, 7, 10] for example). In its general form, (2) requires a numerical resolution. Only simple forms of *f* are tractable. We have given approximate solutions for D_1 and D_3 under mixed mode I+III loading when $h = 0$ and $\xi = \xi(x_3)$ depends on x_3 only [2]. Fortunately, we can give approximate expressions for the stress about the crack front and the crack extension with *f* given by (1), taking for $D_i(x_1)$ dislocation distributions of straight dislocation arrays obtained inSection 3 below; such approximations have been performed in [2, 3]. ***Figure 2*** (taken from [3]) is a schematic representation of simple special cracks captured by the modelling. The cracks extend in the x_1 – direction from $x_1 = -a$ to a and must be considered to run indefinitely in the x_3 – direction.

The crack shape in planes perpendicular to x_1 is described by ξ (***Figure 2c*** for example). The shape f of the crack in planes perpendicular to x_3 is given by both ξ, through the x_1 – dependence of positive quantities ξ_n, δ_n and κ_n (Equation (1)), and function $h = h(x_1)$. Since ξ is assumed to be small oscillating function, the average fracture plane is described correctly by the equation $x_2 = h(x_1)$. When $\xi = 0$, the crack dislocations are straight parallel to x_3 and distributed over the surface $x_2 = h(x_1)$. Specific examples are ***(Figure 2)*** :

- $h(x_1) = p_0 x_1$ ($p_0 \geq 0$) and $\xi = 0$. This corresponds to a planar crack π_0 (with a straight front parallel to x_3) rotated around Ox_3 by angle $\theta_0 = \tan^{-1} p_0$ from Ox_1x_3, ***Figure 2a***.
- $h(x_1)$ is an arbitrary function of x_1 and $\xi = 0$. The sketch in ***Figure 2b*** corresponds to h odd although this is not mandatory. Actually h odd conforms well to homogeneity of the medium, geometry of the applied

loadings and D_i (Section 3) approximations adopted in the present study.

- $h(x_1) = p_0 x_1$ ($p_0 \geq 0$) and $\xi = \xi(x_3)$ independent of x_1. The crack fluctuates about plane π_0 with a front spreading in planes parallel to $x_2 x_3$ in the form ξ. In the example displayed in ***Figure 2c*** the crack consists of planar facets with inclination angles ϕ_A and ϕ_B ***(Figure 2d)*** at points *A* and *B* of the crack front located on the average fracture plane. Points *A* and *B* are indicated in ***Figure 2c***.

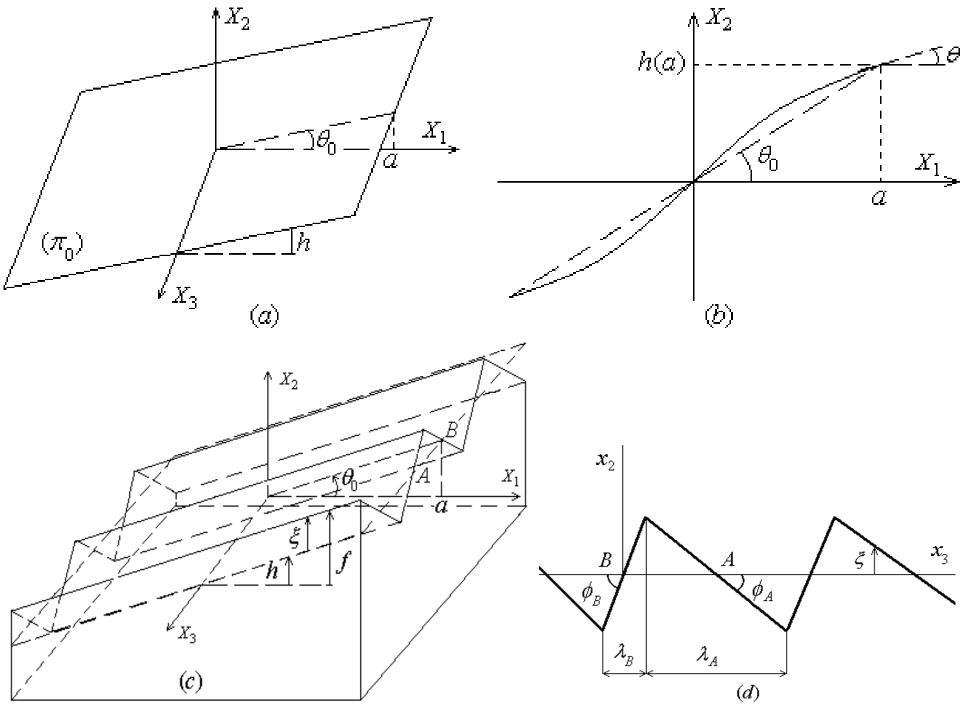

Figure 2 : *Simple special cracks. (a) Inclined planar crack π_0 (see text). (b) A non-planar crack (parallel to x_3) as h odd function of x_1 ($x_2 = h(x_1)$). (c) Non-planar crack fluctuating about an average inclined plane π_0. The crack consists of planar facets; its fronts at $x_1 = \pm a$ lie in $x_2 x_3$ – planes. At $x_1 = a$, the crack front is characterized by inclination angles ϕ_A and ϕ_B (see (d)) at points A and B located on the average fracture plane. (d) Sketch of the crack front in (c) with B taken as origin. In this geometry (from (a) to (c)) the general loading of the crack systems corresponds to uniform applied σ_{22}^a, σ_{12}^a and σ_{23}^a at infinity in the x_2, x_1 and x_3 directions, respectively*

III - RESULTS

III-1. Dislocation distributions

Assume first that the dislocations are straight parallel to the x_3 – direction $(\xi = 0)$ and $h(x_1) = p_0 x_1$ depends linearly on x_1 with p_0 positive constant. This corresponds to a planar crack (π_0 in ***Figure 2a***) of finite extension, with straight fronts running indefinitely along x_3, rotated (from Ox_1x_3) about the positive x_3 – direction by $\theta_0 = \tan^{-1} p_0$. The crack extends from $x_1 = -a$ to a and is subjected to mixed mode I+II+III with loadings applied at infinity. Under such conditions, we have $\partial f / \partial x_1 = \partial h / \partial x_1 = p_0$, $\partial f / \partial x_3 = \partial \xi / \partial x_3 = 0$; the condition (2) for the crack faces to be free from tractions becomes

$$\begin{cases} \sigma_{22}^a - p_0 \sigma_{12}^a + C_1 \int_{-a}^{a} \dfrac{D_1(x_1')}{x_1 - x_1'} dx_1' = 0 \\ \sigma_{12}^a + p_0 \nu_A \sigma_{22}^a + C_1 \int_{-a}^{a} \dfrac{D_2(x_1')}{x_1 - x_1'} dx_1' = 0 \\ \sigma_{23}^a + C_2 \int_{-a}^{a} \dfrac{D_3(x_1')}{x_1 - x_1'} dx_1' = 0 \end{cases} \tag{5}$$

where the Cauchy principal values of the integrals must be taken; $C_1 = \mu b / 2\pi(1-\nu)$, $C_2 = \mu b / 2\pi$ and ν_A is the Poisson ratio, so denoted, to identify the contributions of the normal induced stresses due to the Poisson effect under consideration. The type of solution is well known [11] :

$$D_1(x_1) = \left(1 - p_0 \frac{\sigma_{12}^a}{\sigma_{22}^a}\right) \frac{\sigma_{22}^a x_1}{\pi C_1 \sqrt{a^2 - x_1^2}} \equiv \left(1 - p_0 \frac{\sigma_{12}^a}{\sigma_{22}^a}\right) D_0^{(I)}(x_1),$$

$$D_2(x_1) = \left(1 + p_0 \frac{\nu_A \sigma_{22}^a}{\sigma_{12}^a}\right) \frac{\sigma_{12}^a x_1}{\pi C_1 \sqrt{a^2 - x_1^2}} \equiv \left(1 + p_0 \frac{\nu_A \sigma_{22}^a}{\sigma_{12}^a}\right) D_0^{(II)}(x_1),$$

$$D_3(x_1) = \frac{\sigma_{23}^a}{\pi C_2} \frac{x_1}{\sqrt{a^2 - x_1^2}} \equiv D_0^{(III)}(x_1); \tag{6}$$

$D_0^{(i)}$ ($i = I$, *II* and *III* respectively) corresponds to the equilibrium distribution of straight dislocations when the crack is planar in the Ox_1x_3 – plane ($p_0 = 0$), extending from $x_1 = -a$ to a , under pure mode i loading. The corresponding relative displacement ϕ_i of the crack faces, in the x_2 ($i = 1$), x_1 ($i = 2$) and x_3 ($i = 3$) directions, are :

$$\phi_1(x_1) = \left(1 - p_0 \frac{\sigma_{12}^a}{\sigma_{22}^a}\right) \frac{b\sigma_{22}^a}{\pi C_1} \sqrt{a^2 - x_1^2} \equiv \left(1 - p_0 \frac{\sigma_{12}^a}{\sigma_{22}^a}\right) \phi_0^{(I)}(x_1) ,$$

$$\phi_2(x_1) = \left(1 + p_0 \frac{\nu_A \sigma_{22}^a}{\sigma_{12}^a}\right) \frac{b\sigma_{12}^a}{\pi C_1} \sqrt{a^2 - x_1^2} \equiv \left(1 + p_0 \frac{\nu_A \sigma_{22}^a}{\sigma_{12}^a}\right) \phi_0^{(II)}(x_1) ,$$

$$\phi_3(x_1) = \frac{b\sigma_{23}^a}{\pi C_2} \sqrt{a^2 - x_1^2} \equiv \phi_0^{(III)}(x_1) ; \tag{7}$$

$\phi_0^{(i)}$ ($i = I$, *II* and *III* respectively), similarly as $D_0^{(i)}$, corresponds to the relative displacement of the crack faces when the crack is in Ox_1x_3 under pure mode i loading. D_i is unbounded at $x_1 = \pm a$ and the ϕ_i curve vertical at these end points.

III-2. Stresses about the crack front

To obtain expressions for the stress about the crack front, we proceed as follows. In the neighbourhood of the crack front located at $x_1 = a$, any point P with coordinates (x_1, x_2, x_3) is characterized by x_2 close to h since the fracture surface is given by $f = h + \xi$ with ξ small. We can thus consider the Taylor series expansion of $\sigma_{ij}^{(n)}(x_1 - x_1', x_2, x_3)$ about $x_2 = h(x_1)$ to first order with respect to $(x_2 - h)$; this gives

$$\sigma_{ij}^{(n)}(x_1 - x_1', x_2, x_3) = \sigma_{ij}^{(n)}(x_1 - x_1', h, x_3) + \frac{\partial \sigma_{ij}^{(n)}}{\partial x_2}(x_2 - h) + o(x_2 - h) \tag{8}$$

where $o(x_2 - h)$ is the complementary part of the series form. Writing $x_1 = a + s$, $0 < s \ll a$, $\bar{\sigma}_{ij}$ (3) is given by the following formula:

$$\bar{\sigma}_{ij}(s, x_2, x_3) = \sum_{n=1}^{3} \int_{a-\delta a}^{a} \sigma_{ij}^{(n)}(a + s - x_1', x_2, x_3) D_n(x_1') dx_1' \tag{9}$$

with $\delta a \ll a$. This stress expression means that only those dislocations located about the crack front in x_1 – interval $[a - \delta a \;,\; a]$ will contribute significantly to the stress at $x_1 = a + s$ ahead of the crack tip as s tends to zero; any other contribution will be negligible for a sufficiently small value of s. We observe that this formula is precise with no place for any other kind of additional stress term. Applying the Taylor expansion (8), in $\sigma_{ij}^{(n)}(x_1 - x_1^{'}, h(x_1), x_3)$ and $\partial\sigma_{ij}^{(n)} / \partial x_2$ (in which $x_1 = a + s$), appears the difference ($h(x_1) - h(x_1^{'})$) which we express as follows since x_1 and $x_1^{'}$ (see (9)) are close to a : $h(x_1) = h(a) + p(x_1 - a) + o(x_1 - a)$ and $h(x_1^{'}) = h(a) + p(x_1^{'} - a) + o(x_1^{'} - a)$ where $p = \partial h(a) / \partial x_1$; therefore $h(x_1) - h(x_1^{'}) = p(x_1 - x_1^{'}) + o(x_1 - x_1^{'})$. Furthermore in $\overline{\sigma}_{ij}$ (9) we restrict ourselves to singularities of the type $s^{-1/2}$ only; this is the singularity that comes into play in the study of planar cracks and gives a well-defined value to the crack extension force. This corresponds to identify $\sigma_{ij}^{(n)}$ to the unbounded terms with $1/(a + s - x_1')$ in the Taylor expansion (8). Assuming $\xi(x_1, x_3)$ and its spatial derivatives with respect to x_3 be bounded at $x_1 = a$, the involved integrals in (9) are of the type $\int D_n(x_1')/(a + s - x_1')dx_1'$ which is calculated approximately taking for D_n the straight edge and screw dislocation distributions (6) corresponding to a planar crack π_0 with a straight front parallel to x_3 ***(Figure 2a)***. We obtain ($\overline{\sigma}_{ij} \equiv \overline{\sigma}_{ij}^{(1)} + \overline{\sigma}_{ij}^{(2)} + \overline{\sigma}_{ij}^{(3)}$) :

$$\overline{\sigma}_{ii}^{(1)}(s, x_2, x_3) = \frac{1}{(1 + p^2)^3}\Big([\delta_{i1} + \delta_{i2} + 2\nu\delta_{i3} + (-\delta_{i1} + 3\delta_{i2} + 2\nu\delta_{i3})p^2](1 + p^2)$$

$$+ \frac{1}{2}(x_2 - h(a))\big[\delta_{i1} - \delta_{i2} + 2(1+\nu)\delta_{i3} - 6(\delta_{i1} - \delta_{i2})p^2$$

$$- (-\delta_{i1} + \delta_{i2} + 2(1+\nu)\delta_{i3})p^4\big]\frac{\partial^2\xi}{\partial x_3^2}(a, x_3)\Big)\left(1 - p_0\frac{\sigma_{12}^a}{\sigma_{22}^a}\right)\frac{K_I^0}{\sqrt{2\pi}}\frac{1}{\sqrt{s}},$$

$$\overline{\sigma}_{ii}^{(2)}(s, x_2, x_3) = \frac{(-\delta_{i1} + \delta_{i2} - \delta_{i3})p}{(1 + p^2)^3}\Big([3\delta_{i1} + \delta_{i2} + 2\nu\delta_{i3} + (\delta_{i1} - \delta_{i2} + 2\nu\delta_{i3})p^2](1 + p^2)$$

$$+ \frac{1}{2}(x_2 - h(a))\big[3\delta_{i1} + (3 + 4\nu)\delta_{i2} + 2(1 + 3\nu)\delta_{i3} + (-6\delta_{i1} - 2(3 - 4\nu)\delta_{i2} + 8\nu\delta_{i3})p^2$$

$$+ (-\delta_{i1} - (1 - 4\nu)\delta_{i2} - 2(1-\nu)\delta_{i3})p^4\big]\frac{\partial^2\xi}{\partial x_3^2}(a, x_3)\Big)\left(1 + p_0\frac{\nu_A\sigma_{22}^a}{\sigma_{12}^a}\right)\frac{K_{II}^0}{\sqrt{2\pi}}\frac{1}{\sqrt{s}},$$

$$\overline{\sigma}_{ii}^{(3)}(s, x_2, x_3) = \frac{\left[(p^2 - 1)\delta_{i1} + \left(1 - 2\nu - (1 + 2\nu)p^2\right)\delta_{i2} - 2(1 + p^2)\delta_{i3}\right]}{(1-\nu)(1 + p^2)^2}\frac{\partial\xi}{\partial x_3}\frac{K_{III}^0}{\sqrt{2\pi}}\frac{1}{\sqrt{s}},$$

$$\overline{\sigma}_{j3}^{(1)}(s,x_2,x_3)=\frac{p[-(2-\nu)+\nu p^2]\delta_{j1}+[\nu-(2-\nu)p^2]\delta_{j2}}{(1+p^2)^2}\frac{\partial\xi}{\partial x_3}\left(1-p_0\frac{\sigma_{12}^a}{\sigma_{22}^a}\right)\frac{K_I^0}{\sqrt{2\pi}}\frac{1}{\sqrt{s}},$$

$$\overline{\sigma}_{j3}^{(2)}(s,x_2,x_3)=-\frac{[1-(1-\nu)p^2-(2-\nu)p^4]\delta_{j1}+p[1+2\nu-(1-2\nu)p^2]\delta_{j2}}{(1+p^2)^2}$$

$$\times\frac{\partial\xi}{\partial x_3}\left(1+p_0\frac{\nu_A\sigma_{22}^a}{\sigma_{12}^a}\right)\frac{K_{II}^0}{\sqrt{2\pi}}\frac{1}{\sqrt{s}},$$

$$\overline{\sigma}_{j3}^{(3)}(s,x_2,x_3)=\frac{1}{(1-\nu)(1+p^2)^2}\left((1-\nu)[-p\delta_{j1}+\delta_{j2}](1+p^2)-\frac{1}{2}(x_2-h(a))\right.$$

$$\left.\times\left[p\left(5-3\nu-(1+\nu)p^2\right)\delta_{j1}-\left(3-\nu-(3-\nu)p^2\right)\delta_{j2}\right]\frac{\partial^2\xi}{\partial x_3^2}\right)\frac{K_{III}^0}{\sqrt{2\pi}}\frac{1}{\sqrt{s}},$$

$$\overline{\sigma}_{12}^{(1)}(s,x_2,x_3)=\frac{p}{(1+p^2)^3}\left(1-p^4+\frac{1}{2}(x_2-h(a))\right.$$

$$\left.\times\left[5-2\nu-2(1+2\nu)p^2+(1-2\nu)p^4\right]\frac{\partial^2\xi}{\partial x_3^2}\right)\left(1-p_0\frac{\sigma_{12}^a}{\sigma_{22}^a}\right)\frac{K_I^0}{\sqrt{2\pi}}\frac{1}{\sqrt{s}},$$

$$\overline{\sigma}_{12}^{(2)}=\frac{1}{(1+p^2)^3}\left(1-p^4+\frac{1}{2}(x_2-h(a))\left[1+2(13+2\nu)p^2+p^4\right]\frac{\partial^2\xi}{\partial x_3^2}\right)$$

$$\times\left(1+p_0\frac{\nu_A\sigma_{22}^a}{\sigma_{12}^a}\right)\frac{K_{II}^0}{\sqrt{2\pi}}\frac{1}{\sqrt{s}},$$

$$\overline{\sigma}_{12}^{(3)}(s,x_2,x_3)=-\frac{p(2-\nu-\nu p^2)}{(1-\nu)(1+p^2)^2}\frac{\partial\xi}{\partial x_3}\frac{K_{III}^0}{\sqrt{2\pi}}\frac{1}{\sqrt{s}}\qquad(10)$$

where subscript i and j take the values (1, 2 and 3) and (1 and 2) respectively; $p=\partial h(a)/\partial x_1$, K_i^0 (i = I, II and III respectively) is the SIF for the planar crack in Ox_1x_3 at the origin under pure mode i loading; $K_I^0=\sigma_{22}^a\sqrt{a\pi}$, $K_{II}^0=\sigma_{12}^a\sqrt{a\pi}$ and $K_{III}^0=\sigma_{23}^a\sqrt{a\pi}$. We stress again that s, x_2 and x_3 are arbitrary, $s=x_1-a<<a$ ($s>0$) and $(x_2-h(a))$ is small. The parameter p_0 in (10) originates from a planar crack π_0 ***(Figure 2a)*** hypothetically assumed to approximate the average fracture surface $x_2=h(x_1)$. This suggests that we could write $h(x_1)=p_0x_1+\Delta h(x_1)$ where Δh is an oscillating function of x_1 taking small positive and negative values. Taking (6) for D_n results in coefficients $(1-p_0\sigma_{12}^a/\sigma_{22}^a)$, $(1+p_0\nu_A\sigma_{22}^a/\sigma_{12}^a)$ and $K_i^0/\sqrt{2\pi s}$ only in (10), the other factors have no concern with this approximation.

III-3. Crack extension force

In the following, an expression for the derivative G of the energy of the system with respect to crack area is derived. This serves to discuss the initiation of crack motion and provide an expression for the crack extension force. We follow [1 to 3] and the procedure is adapted from [11]. Allow the right-hand front of the non-planar crack with shape (1) (use ***Figure 2c*** to illustrate) to advance (say rigidly for simplicity) from $x_1 = a$ to $a + \delta a$, but apply forces to the freshly formed surfaces to prevent relative displacement; the energy of the system is unaltered. Now allow these forces to relax to zero so that the crack extends effectively from a to $a + \delta a$. The work done by these forces corresponds to a decrease of the energy of the system which we shall estimate (the energy of the system consists of the elastic energy of the medium and the energy of the loading mechanism). The element ds of the fracture surface $x_2 = f(x_1, x_3)$ ahead of the crack front, at a point $P = (x_1, x_2 = f, x_3)$, may be defined by $d\vec{s} = \vec{\gamma} ds$ where $\vec{\gamma}$ is the unit vector perpendicular to ds pointing to the positive x_2 – direction.

We obtain $d\vec{s} = 1/\sqrt{1 + (\partial f/\partial x_1)^2 + (\partial f/\partial x_3)^2}\,(-\partial f/\partial x_1, 1, -\partial f/\partial x_3)ds$ and take ds to be given by $ds = \sqrt{1 + (\partial f/\partial x_1)^2 + (\partial f/\partial x_3)^2}\,dx_1 dx_3$. The component of the force acting on ds in the x_i – direction is $\bar{\sigma}_{ij} ds_j$ (the summation convention on repeated subscripts applies) where $\bar{\sigma}_{ij}$ are stresses ahead of the shorter crack; thus the energy change associated with ds is $\bar{\sigma}_{ij} ds_j \Delta u^{(i)}/2$ (here a summation is also considered over i = 1, 2 and 3) where $\Delta u^{(i)}$ is the difference in displacement across the lengthened crack, just behind its tip, in the x_i – direction. When the crack advances from $x_1 = a$ to $a + \delta a$, the energy decrease associated with a surface element

$$\Delta s = \int_a^{a+\delta a} ds \cong \sqrt{1 + (\partial f/\partial x_1)^2 + (\partial f/\partial x_3)^2}\,\delta a dx_3$$

(δa being small and, when used below, will be let to go to zero) is given as

$$-\delta E = \frac{1}{2}\int_a^{a+\delta a} \sum_i \sum_j \bar{\sigma}_{ij} ds_j \Delta u^{(i)},$$

the integration being performed with respect to x_1 ; we stress that Δs is the sum of the surface elements ds taken at the various points $P = (x_1, x_2 = f, x_3)$ as x_1 only changes from a to $a + \delta a$. Let G be a derivative of the energy of the system with respect to crack area. G corresponds to the limiting value taken by $-\delta E / \Delta s$ as δa (as also Δs) decreases to zero. Stresses $\bar{\sigma}_{ij}$ generally consist of terms that are either bounded or unbounded as x_1 tends to a ; only those stress terms that are singular may contribute a non-zero value to G; the bounded terms all contribute nothing. Using (10) and defining $\bar{\sigma}^{(i)}(s)$ ($i = 1$, 2 and 3) as

$$\bar{\sigma}^{(i)}(s) \equiv \frac{\delta_{i1} K_{II}^{0} + \delta_{i2} K_{I}^{0} + \delta_{i3} K_{III}^{0}}{\sqrt{2\pi}} \frac{1}{\sqrt{s}} \tag{11}$$

we arrive at

$$G(P_0) = \lim_{\delta a \to 0} -\delta E / \Delta s$$
$$= \frac{1}{\sqrt{1 + (\partial f / \partial x_1)^2 + (\partial f / \partial x_3)^2}} \left([\circ] G_0^{(1)} + [\circ\,\circ] G_0^{(2)} + [\circ\,\circ\,\circ] G_0^{(3)} \right) \tag{12}$$

where

$$G_0^{(i)} = \lim_{\delta a \to 0} \frac{1}{\delta a} \left(\frac{1}{2} \int_a^{a+\delta a} \bar{\sigma}^{(i)} \Delta u^{(i)} dx_1 \right), \; i = 1, 2 \text{ and } 3. \tag{13}$$

Expressions in $[\circ]$, $[\circ\,\circ]$ and $[\circ\,\circ\,\circ]$ are very long and need not be displayed here; an explicit presentation (16) follows. (12) gives the value of G at an arbitrary point $P_0(a, x_2 = f, x_3)$ along the front of the non-planar crack with projected half-length a along x_1 . The calculation of $\Delta u^{(i)}$ depends on the way the extension of the right-hand front of the crack from $x_1 = a$ to $a + \delta a$ is performed. When $\Delta u^{(i)}$ is obtained from a distribution of dislocations perpendicular to the x_1 -direction, we implicitly assume a rigid crack-front displacement (i.e. the crack front moves rigidly). In that case, $\Delta u^{(i)}$ may be obtained from the solution of (2) modified to allow for the fact that the crack extends from $x_1 = -a$ to $a + \delta a$ instead of from $-a$ to a . Approximate expressions for $G_0^{(i)}$ ($i = 1$, 2 and 3) correspond to a planar distribution of straight edge and screw dislocations. When the crack has the geometry of

Figure 2a with $\theta_0 = 0$ ($p_0 = 0$), Bilby and Eshelby [11] have shown that $G_0^{(1)} = K_{II}^{0\,2}(1-\nu^2)/E \equiv G_0^{II}$, $G_0^{(2)} = K_I^{0\,2}(1-\nu^2)/E \equiv G_0^I$ and $G_0^{(3)} = K_{III}^{0\,2}(1+\nu)/E \equiv G_0^{III}$ where E is Young's modulus. The corresponding dislocation distributions are $D_0^{(II)}$, $D_0^{(I)}$ and $D_0^{(III)}$ (see (6)) with associated relative displacements of the faces of the crack $\phi_0^{(II)}$, $\phi_0^{(I)}$ and $\phi_0^{(III)}$ (7). In the same approximation and using the dislocation distributions D_i (6) corresponding to a planar crack π_0 ***(Figure 2a)*** inclined by θ_0 (about axis Ox_3) with respect to Ox_1x_3, we arrive at

$$G_0^{(1)} = \left(1 + p_0 \frac{\nu_A \sigma_{22}^a}{\sigma_{12}^a}\right) G_0^{II},$$

$$G_0^{(2)} = \left(1 - p_0 \frac{\sigma_{12}^a}{\sigma_{22}^a}\right) G_0^{I},$$

$$G_0^{(3)} = G_0^{III}. \qquad (14)$$

Adopting approximation (14) and defining $\tilde{G}(P_0)$ as $\tilde{G}(P_0) = G(P_0)/(G_0^I + G_0^{II} + G_0^{III})$, we obtain (with $M_{12} \equiv \sigma_{12}^a/\sigma_{22}^a$, $M_{13} \equiv \sigma_{23}^a/\sigma_{22}^a$, $M_{23} \equiv \sigma_{23}^a/\sigma_{12}^a$)

$$\tilde{G}(P_0) = \sum_{i,j=1}^{3} \tilde{G}_j^{(i)}(P_0) \qquad (15)$$

where

$$\tilde{G}_1^{(1)} = -\frac{1}{2(1+p^2)^3} \frac{\partial f(a,x_3)/\partial x_1}{\sqrt{1+(\partial f/\partial x_1)^2+(\partial f/\partial x_3)^2}} \left(2(1-p^4)(1-p_0 M_{12}) - 2p(1+p^2) \right.$$

$$\times (3+p^2)(p_0\nu_A + M_{12}) + \frac{2}{1-\nu}(p^4-1)M_{13}\frac{\partial \xi}{\partial x_3} + \left[(1-6p^2+p^4)(1-p_0 M_{12})\right.$$

$$\left.\left. - p(3-6p^2-p^4)\right)(p_0\nu_A + M_{12})\right]\xi \frac{\partial^2 \xi}{\partial x_3^2}\right) \frac{(1-\nu)(p_0\nu_A + M_{12})}{\left[1-\nu+(1-\nu)M_{12}^2 + M_{13}^2\right]},$$

$$\tilde{G}_2^{(1)} = \frac{1}{2(1+p^2)^3}\frac{1}{\sqrt{1+(\partial f/\partial x_1)^2+(\partial f/\partial x_3)^2}}\Bigg(2(1-p^4)[p(1-p_0M_{12})+M_{12}+p_0\nu_A]$$

$$-\frac{2}{1-\nu}p(1+p^2)(2-\nu-\nu p^2)M_{13}\frac{\partial\xi}{\partial x_3}+\left[p\left(5-2\nu-2(1+2\nu)p^2+(1-2\nu)p^4\right)\right.$$

$$\left.\times(1-p_0M_{12})+\left(1+2(13+2\nu)p^2+p^4\right)(M_{12}+p_0\nu_A)\right]$$

$$\times\xi\frac{\partial^2\xi}{\partial x_3^2}\Bigg)\frac{(1-\nu)(p_0\nu_A+M_{12})}{\left[1-\nu+(1-\nu)M_{12}^2+M_{13}^2\right]},$$

$$\tilde{G}_3^{(1)} = -\frac{1}{(1+p^2)^2}\frac{\partial f/\partial x_3}{\sqrt{1+(\partial f/\partial x_1)^2+(\partial f/\partial x_3)^2}}\Bigg(-p(1+p^2)M_{13}+\left[p\left(-2+\nu+\nu p^2\right)\right.$$

$$\left.\times(1-p_0M_{12})-\left(1-(1-\nu)p^2-(2-\nu)p^4\right)(p_0\nu_A+M_{12})\right]\frac{\partial\xi}{\partial x_3}$$

$$-p\left(5-3\nu-(1+\nu)p^2\right)\frac{M_{13}}{2(1-\nu)}\xi\frac{\partial^2\xi}{\partial x_3^2}\Bigg)\frac{(1-\nu)(p_0\nu_A+M_{12})}{\left[1-\nu+(1-\nu)M_{12}^2+M_{13}^2\right]},$$

$$\tilde{G}_1^{(2)} = -\tilde{G}_2^{(1)}\frac{\partial f}{\partial x_1}\frac{1-p_0M_{12}}{p_0\nu_A+M_{12}},$$

$$\tilde{G}_2^{(2)} = \frac{1}{2(1+p^2)^3}\frac{1}{\sqrt{1+(\partial f/\partial x_1)^2+(\partial f/\partial x_3)^2}}\Bigg(2(1+p^2)\left[(1+3p^2)(1-p_0M_{12})\right.$$

$$\left.+p(1-p^2)(M_{12}+p_0\nu_A)\right]+\frac{2}{1-\nu}(1+p^2)(1-2\nu-(1+2\nu)p^2)M_{13}\frac{\partial\xi}{\partial x_3}$$

$$+\left[(-1+6p^2-p^4)(1-p_0M_{12})+p\left(3+4\nu-2(3-4\nu)p^2-(1-4\nu)p^4\right)(M_{12}+p_0\nu_A)\right]$$

$$\times\xi\frac{\partial^2\xi}{\partial x_3^2}\Bigg)\frac{(1-\nu)(1-p_0M_{12})}{\left[1-\nu+(1-\nu)M_{12}^2+M_{13}^2\right]},$$

$$\tilde{G}_3^{(2)} = -\frac{1}{(1+p^2)^2}\frac{\partial f/\partial x_3}{\sqrt{1+(\partial f/\partial x_1)^2+(\partial f/\partial x_3)^2}}\Bigg((1+p^2)M_{13}+\Bigg[\left(\nu-(2-\nu)p^2\right)$$

$$\times(1-p_0M_{12})-p\left(1+2\nu-(1-2\nu)p^2\right)(M_{12}+p_0\nu_A)\Bigg]\frac{\partial\xi}{\partial x_3}+\frac{(3-\nu)(1-p^2)M_{13}}{2(1-\nu)}$$

$$\times\xi\frac{\partial^2\xi}{\partial x_3^2}\Bigg)\frac{(1-\nu)(1-p_0M_{12})}{\left[1-\nu+(1-\nu)M_{12}^2+M_{13}^2\right]},$$

$$\tilde{G}_1^{(3)} = \tilde{G}_3^{(1)} \frac{\partial f / \partial x_1}{\partial f / \partial x_3} \frac{M_{13}}{(1-\nu)(p_0\nu_A + M_{12})},$$

$$\tilde{G}_2^{(3)} = -\tilde{G}_3^{(2)} \frac{1}{\partial f / \partial x_3} \frac{M_{13}}{(1-\nu)(1-p_0 M_{12})},$$

$$\tilde{G}_3^{(3)} = -\frac{1}{2(1+p^2)^3} \frac{\partial f/\partial x_3}{\sqrt{1+(\partial f/\partial x_1)^2+(\partial f/\partial x_3)^2}} \Bigg(4\nu(1+p^2)^2\left[1-(p+p_0)M_{12} - pp_0\nu_A\right]$$

$$-\frac{4}{1-\nu}(1+p^2)^2 M_{13}\frac{\partial \xi}{\partial x_3} + \left[2(1+\nu)(1-p^4)(1-p_0M_{12})\right.$$

$$\left.-p\left(2(1+3\nu)+8\nu p^2 - 2(1-\nu)p^4\right)(M_{12}+p_0\nu_A)\right]\xi\frac{\partial^2\xi}{\partial x_3^2}\Bigg) \frac{M_{13}}{\left[1-\nu+(1-\nu)M_{12}^2+M_{13}^2\right]}. \quad (16)$$

For the planar crack with a straight front, the decrease of the energy of the system ($-\delta E$), divided by the surface element $dl\,\delta a$ (l runs along the crack front), is defined as the crack extension force per unit edge length of the crack front (see, for example, [11]). In the present study, we shall refer to *G* (12) as the crack extension force per unit length of the crack front. In Section 3.4, we give a more detailed description of *G* for special cracks.

III-4. Special cracks

We consider first the crack in ***Figure 2b***; it extends from $x_1 = -a$ to a and runs indefinitely in the x_3 – direction. The crack front is straight parallel to x_3 ($\xi = 0$) and $f = h(x_1)$ independent of x_3. We assume the crack to fluctuate about plane π_0 ***(Figure 2a)*** and take D_i (6) as the distribution of the equilibrium crack dislocations. Under such conditions the reduced crack extension force $\tilde{G}$ (15) under mixed mode I+II+III loading takes the form

$$\tilde{G}(P_0) = \frac{(p_0\nu_A + M_{12})^2 + (1-p_0M_{12})^2 + M_{13}^2/(1-\nu)}{\sqrt{1+p^2}\left[1+M_{12}^2+M_{13}^2/(1-\nu)\right]} \quad (\xi = 0; h = h(x_1)) \quad (17)$$

at point $P_0(a, x_2 = h(a), x_3)$ of the crack front located at $x_1 = a$; $p = \partial h(a)/\partial x_1 = \tan\theta$ (for θ, see ***Figure 2b***). A similar relation (18) below will be given a more detailed description.

The second crack we present is given in ***Figure 2a***. This is a tilted planar crack corresponding to the rotation of plane Ox_1x_3 about Ox_3 by angle $\theta_0 = \tan^{-1}(p_0)$

. The crack front is straight ($\xi = 0$) and $h = p_0 x_1$. The normalized crack extension force $\tilde{G}$ (15) then reads

$$\tilde{G}(P_0) = \frac{(p_0 \nu_A + M_{12})^2 + (1 - p_0 M_{12})^2 + M_{13}^2/(1-\nu)}{\sqrt{1 + p_0^2}\left[1 + M_{12}^2 + M_{13}^2/(1-\nu)\right]} \quad (\xi = 0; h = p_0 x_1) \qquad (18)$$

at point $P_0(a, x_2 = p_0 a, x_3)$ of the crack front located at $x_1 = a$. ***Figures 3 and 4*** are plots of $\tilde{G}$ (18) as a function of θ_0 for different M_{12} values when $M_{13} = 0$ (mixed mode I+II loading) under $\nu_A = 0$ (absence of Poisson effect) and $\nu_A \neq 0$ (presence of Poisson effect), respectively. We expect that for a tilted plane crack π_0 ***(Figure 2a)*** to be observable experimentally, it is necessary that $\tilde{G}$ be larger than 1. As we can see from ***Figure 3***, this occurs for sufficiently large M_{12} and θ_0 values; the larger the M_{12} (dominant mode II loading) the smaller the θ_0 values for which $\tilde{G} > 1$. As we can see from ***Figure 4***, Poisson effect increases significantly $\tilde{G}$ values.

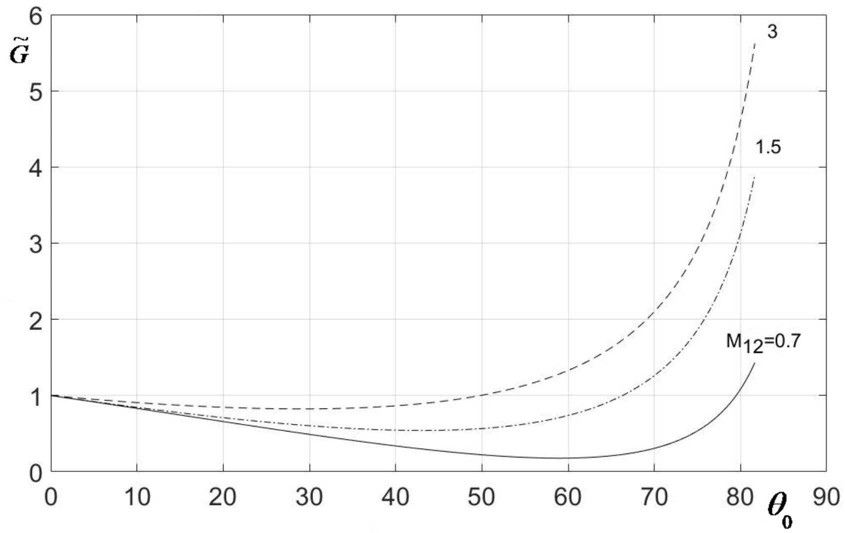

Figure 3 : *Normalized crack extension force $\tilde{G}$ (18) versus θ_0 ($\nu_A = 0$ absence of Poisson effect) for a planar crack π_0 with a straight front parallel to Ox_3 (Figure 2a), tilted about Ox_3 by an angle $\theta_0 = \tan^{-1} p_0$ from Ox_1x_3, under mixed mode I+II loading ($M_{13} = 0$). The curves correspond from bottom to top to $M_{12} = 0.7$, 1.5 and 3*

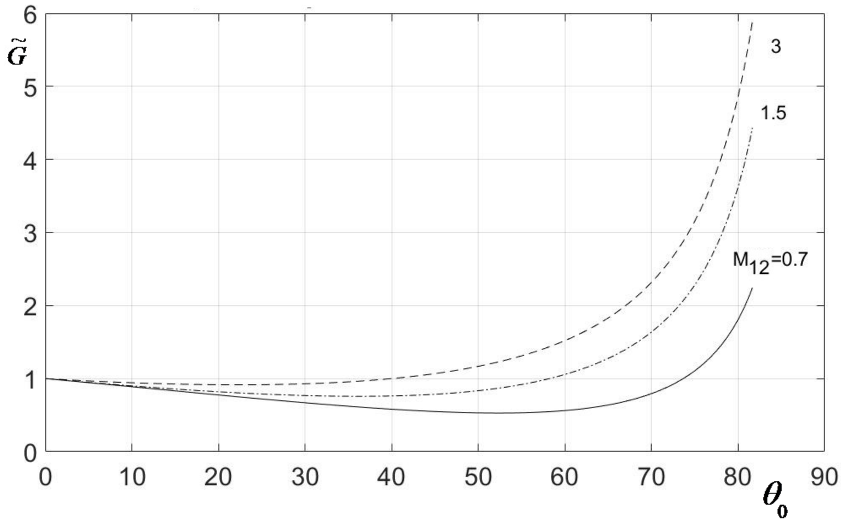

Figure 4 : *Normalized crack extension force $\tilde{G}$ (18) versus θ_0 ($\nu_A \neq 0$ presence of Poisson effect) for a planar crack π_0 (other conditions as in Figure 3)*

The third example we shall describe is given in ***Figure 2c***. This is a non-planar crack with a segmented front (***Figure 2d***) whose average fracture surface is plane π_0 (illustrated in ***Figure 2a***). The crack front at $x_1 = a$ runs indefinitely in the x_3 – direction and is in a $x_2 x_3$ – plane. We describe ξ below taking locally B as origin as in ***Figure 2d***. ξ is then odd and $(2\lambda = \lambda_A + \lambda_B)$ – periodical with respect to x_3 where λ_A and λ_B ***(Figure 2d)*** are the projected length along x_3 of planar facet A and B respectively. ξ is given by :

$$\begin{aligned} \xi &= \tan \phi_B x_3 , & |x_3| \leq \lambda_B / 2 \\ &= \tan \phi_A (-x_3 + \lambda) , & x_3 \in [\lambda_B / 2, \ \lambda_B / 2 + \lambda_A] . \end{aligned} \tag{19}$$

We assume general loading (mixed mode I+II+III), write $p_0 = p = \tan \theta$ for simplicity in (16) for the reduced crack extension force, now denoted $\tilde{G}_v$, and express the spatial average $<\tilde{G}_v>$ of $\tilde{G}_v$ defined as $<\tilde{G}_v> = (1/2\lambda) \int_0^{2\lambda} \tilde{G}_v dx_3$.

We obtain

$$<\tilde{G}_v> = \frac{1}{(1+p^2)^2} \left\{ (1+p^2)^2 \left[M_{13}^2 + (1-\nu) \left((p\nu_A + M_{12})^2 + (1 - pM_{12})^2 \right) \right] \nu_0 \right.$$

$$+\left[p^3\left(\nu-(2-\nu)p^2\right)(p\nu_A+M_{12})-2\nu(1+p^2)^2(1-pM_{12})\right]M_{13}v_1$$
$$+\left[\left\{p\left(3+\nu-p^2(1-\nu)\right)(1-pM_{12})+\left(1-(1-\nu)p^2-(2-\nu)p^4\right)(p\nu_A+M_{12})\right\}\right.$$
$$\left.\times(p\nu_A+M_{12})-\left(\nu-(2-\nu)p^2\right)(1-pM_{12})^2+2(1+p^2)M_{13}^2/(1-\nu)^2\right](1-\nu)v_2\Bigg\}$$
$$\times\frac{1}{(1-\nu)+(1-\nu)M_{12}^2+M_{13}^2} \tag{20}$$

where

$$v_0=\left(1/(p_A+p_B)\right)\left(p_A/\sqrt{1+p^2+p_B^2}+p_B/\sqrt{1+p^2+p_A^2}\right),$$
$$v_1=\left(p_Ap_B/(p_A+p_B)\right)\left(-1/\sqrt{1+p^2+p_A^2}+1/\sqrt{1+p^2+p_B^2}\right),$$
$$v_2=\left(p_Ap_B/(p_A+p_B)\right)\left(p_A/\sqrt{1+p^2+p_A^2}+p_B/\sqrt{1+p^2+p_B^2}\right) \tag{21}$$

and $p_A=\tan\phi_A$, $p_B=\tan\phi_B$. Hence $<\tilde{G}_v>$ is a function of parameters $(p, p_A, p_B; M_{12}, M_{13})$. When ϕ_B (or ϕ_A) equals zero, the crack front is essentially straight parallel to the x_3 – direction. The corresponding crack is like planar crack π_0 ***(Figure 2a)***; under such conditions $<\tilde{G}_v>$ (20) is identical to (18). We have reported on ***Figures 5 to 8*** $<\tilde{G}_v>$ (20) as a function of (θ, ϕ_A) for constant $\phi_B=70°$; ***Figures 5 and 6*** describe mixed mode I+II loading ($M_{13}=0$) and ***Figures 7 and 8*** correspond to mixed mode I+II+III loading, in absence or presence of Poisson effect.

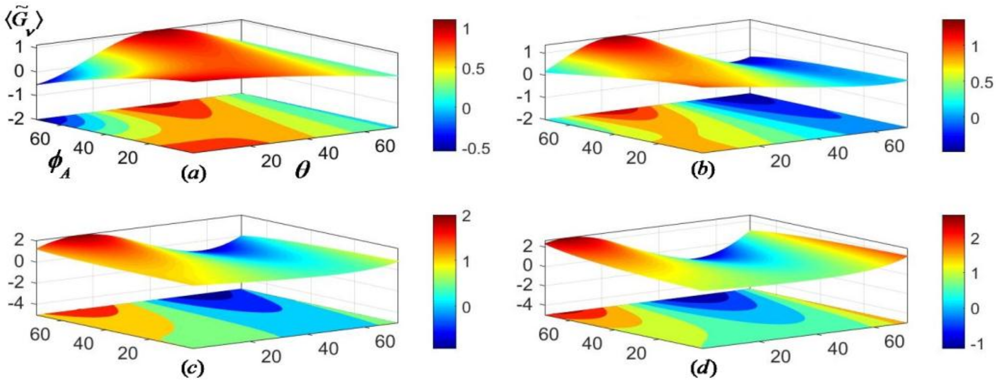

Figure 5 : *Surfaces* $<\tilde{G}_v>(\theta,\phi_A)$ *with associated contours at constant* $\phi_B=70°$, $M_{13}=0$ ($\nu_A=0$ *absence of Poisson effect) and four different* M_{12} *: (a)* $M_{12}=0.1$, *(b) 0.5, (c) 1 and (d) 2 ;* $\nu=1/3$

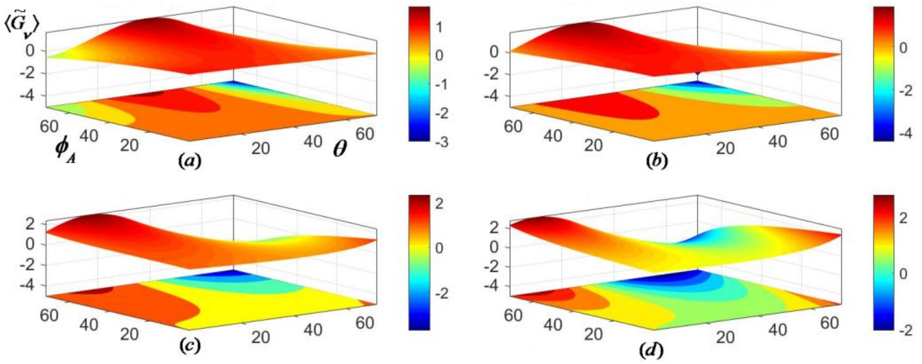

Figure 6 : *Surfaces* $<\tilde{G}_{\nu}>(\theta,\phi_A)$ *with associated contours (*$\nu_A \neq 0$ *presence of Poisson effect; other conditions as in Figure 5)*

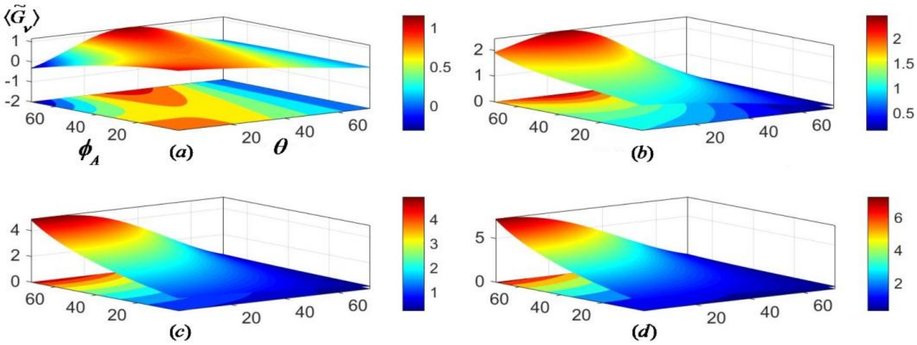

Figure 7 : *Surfaces* $<\tilde{G}_{\nu}>(\theta,\phi_A)$ *with associated contours at constant* $\phi_B = 70°$, $M_{12} = 0.2$ *(*$\nu_A = 0$ *absence of Poisson effect) and four different* M_{13} *: (a)* $M_{13} = 0.1$ *, (b) 0.5, (c) 1 and (d) 2;* $\nu = 1/3$

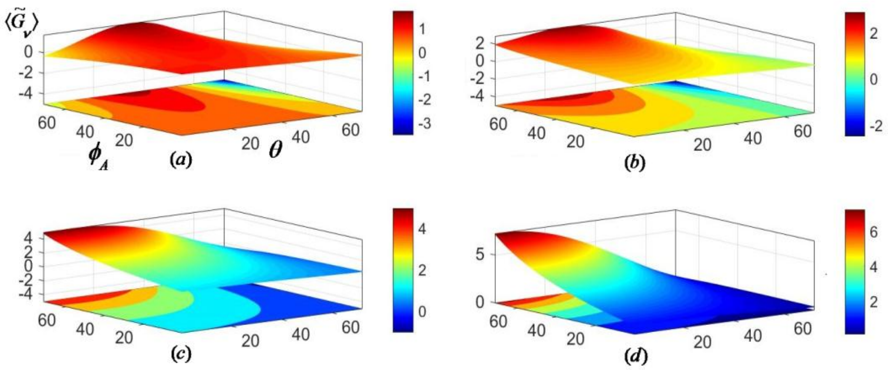

Figure 8 : *Surfaces* $<\tilde{G}_{\nu}>(\theta,\phi_A)$ *with associated contours (*$\nu_A \neq 0$ *presence of Poisson effect; other conditions as in Figure 7)*

IV - DISCUSSION AND CONCLUSION

The normal induced stress $\sigma_{11}^{a} = -\nu\sigma_{22}^{a}$ (originating from Poisson effect) increases clearly the crack extension force for the planar crack tilted around x_3 by angle θ_0 with respect to Ox_1x_3. This is observed on ***Figures 3 and 4*** for mixed mode I+II loading. For $M_{12} = 3$, from values less than 1 and lower angle θ_0 values, $\tilde{G}$ passes above 1 at $\theta_0 = 40°$ in presence of normal stresses ***(Figure 4)***, while in their absence ***(Figure 3)***, this is only done from $\theta_0 = 50°$. The value $\tilde{G} = 1$ corresponds to that of the planar crack in Ox_1x_3. Higher values of $\tilde{G}$ (larger than the value 1 as observed under certain conditions, in ***Figure 4*** for example) means that inclined cracks π_0 ***(Figure 2a)*** are in favourable conditions to move, hence to exist in the broken specimen. The possibility that $\tilde{G}$ may achieved values larger than 1 agrees with experiments showing that when a mode II loading (in addition to mode I) is applied to a planar crack located in Ox_1x_3, the subsequent fracture propagation path departs from x_1x_3 [12 to 14]. In general, the normal stresses induced by the Poisson effect increase the force of extension of the cracks.

This is very clear in ***Figure 6*** compared with ***Figure 5*** for all values of M_{12}. ***Figures 7 and 8*** are also in agreement with this conclusion. Confrontations of our results with other previous analyses have been performed (see [2, 3, 5] for the homogeneous material and [8, 9] for the interface crack, for example). Here, we want to emphasize the agreement of our relationship (18) with previous work [15]. We neglect Poisson effect $\nu_A = 0$ and consider a planar crack in Ox_1x_3 whose shape about one tip is in the form of an infinitesimal kink inclined by angle θ_0 with respect to Ox_1x_3. We assume mixed mode I+II loading and a straight crack front parallel to x_3. Cotterell and Rice [15] have established an expression for the stress intensity factors at the tip of the infinitesimal branch (i.e. their relation (31)) valid for small θ_0. Since their formula does not depend on the shape of the crack, it applies also to an inclined crack π_0 ***(Figure 2a)*** as demonstrated below. The stress intensity factors given in [15] can be expanded to second order in θ_0; these combined with Irwin's expression (i.e. the usual plane strain relation) lead to the following formula for the crack extension force :

$$G = \left(1 - \frac{\theta_0^2}{2} - 2\theta_0 M_{12} + \left(1 + \frac{\theta_0^2}{2}\right) M_{12}^2\right) G_0^I$$

with G_0^I given in the text above (14). It is easy to show that in absence of mode III loading ($M_{13} = 0$), our relation (18) developed to second order in small $p_0 \equiv \theta_0$ gives an identical *G* value. This suggests some confidence in the results (15, 16) displayed in the present study.

In conclusion, the present study provides expressions of crack-tip stresses and crack extension force for non-planar cracks of arbitrary shapes and macroscopic sizes, corresponding to cracking over large distances under arbitrary external loading (mode I+II+III). These expressions incorporate induced normal stresses that result from the contraction, due to Poisson effect, perpendicular to the direction of applied tension. General expressions for the extension force *G* of the crack are sought which consider all the stresses in the material under load. This make it possible to best compare to the experiment, the idea that crack configurations that maximize *G* are those observed on broken materials.

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