ISSN 1813-3290, http://www.revist.ci

A THEORY OF THE FRACTURE OF RECTANGULAR BARS BENT BY TERMINAL TRANSVERSE LOAD AND COUPLE

P. N. B. ANONGBA

UFR Sciences des Structures de la Matière et de Technologie, Université FHB de Cocody, 22 BP 582 Abidjan 22, Côte d'Ivoire

____

* Correspondance, e-mail : *anongba@yahoo.fr*

ABSTRACT

In the present study, an analysis is made of the conditions of failure in a rectangular bar bent by terminal transverse force *F* and terminal couple M_c. The specimen under load consists of a region (1) in tension, separated from a region (2) in compression by a median surface (initially a plane called the "neutral plane") with zero-change in extension or contraction of linear elements. The analysis uses the idea that this is the condition $G_2 = 2\gamma$ that controls the complete failure of the fractured specimen, where G_2 is the crack extension force (per unit length of the crack front) pertaining to the tip of the crack that moves through the compression region (2) and γ, the surface energy. G_2 is calculated in a model with a crack inclined by an angle θ_2 with respect to the local compression direction. The considered stresses are the applied resultant compression $(-\sigma_n)$, the associated induced internal Poisson tension $\nu\sigma_n$ (ν is Poisson's ratio) and the applied shear stress $\tau_{12} = F / S_0$ (S_0 is the cross section of the sample). A representation of the crack by a continuous distribution of straight edge dislocations, with infinitesimal Burgers vectors, is adopted. The theory indicates that when the specimen is subjected to a shear stress (this occurs with a terminal transverse applied load), the propagating crack evolves in the form of an exponential with a bending moment M_f that depends on the elastic constants and specimen dimensions. However, in absence of shear stresses (bending by terminal couple only), the crack tends to align along the local compression direction with a bending moment $M_c = M_f / \nu$ indicating, in the latter situation, that three times larger bending moments, at least, are required to break specimens in isotropic media. A relation between failure stresses in bending and tension tests are also evidenced in the results.

Keywords : *elasticity, fracture mechanics, dislocations, bending test, crack propagation, failure stresses.*

RÉSUMÉ

Une théorie de la rupture d'une barre rectangulaire en flexion sous l'effet d'une charge et d'un couple

Dans la présente étude, une analyse est faite des conditions de rupture dans une barre rectangulaire fléchie par une force verticale *F* et par un couple M_c. L'échantillon sous charge comprend une région (1) sollicitée en tension, séparée d'une région (2) sollicitée en compression, par un plan médian non déformé dit "neutre". La théorie exploite l'idée que c'est la condition $G_2 = 2\gamma$ qui contrôle la rupture complète de l'éprouvette, où G_2 est la force d'extension (par unité de longueur du front de fissure) de l'extrémité de la fissure qui se propage dans la région (2) et γ, l'énergie de surface. G_2 est calculée dans un modèle de fissure inclinée d'un angle θ_2 par rapport à la direction de la compression locale appliquée. Les contraintes en présence sont la contrainte de compression $(-\sigma_n)$, la tension induite correspondante de Poisson $\nu\sigma_n$ (ν est le rapport de Poisson) et la contrainte de cisaillement $\tau_{12} = F / S_0$ (S_0 est la section droite de l'éprouvette). Une représentation de la fissure par une distribution continue de dislocations coins droites, avec des vecteurs de Burgers infinitésimaux, est adoptée. L'analyse indique que lorsque l'échantillon est le siège d'une contrainte de cisaillement (cas de la flexion par une force "transverse"), la fissure évolue dans la forme d'une exponentielle avec un moment de flexion M_f dépendant des dimensions de l'éprouvette et des constantes élastiques. Cependant, en absence de cisaillement (cas de la flexion par couple), la fissure tend à s'aligner suivant la direction locale de la contrainte de compression, avec un moment de flexion $M_c = M_f / \nu$; ce qui indique, dans cette dernière situation, qu'un moment de contrainte trois fois plus grand est nécessaire pour rompre une éprouvette de flexion dans les milieux isotropes. Une relation entre les contraintes à rupture en flexion et en tension découle également des résultats.

Mots-clés : *élasticité, mécanique de la rupture, dislocations, essais de flexion, propagation de fissure, contraintes de rupture*.

I - INTRODUCTION

Consider a rectangular bar ***(Figure 1)*** of length L_0, embedded in a wall at one end in a horizontal position and subjected to a downward vertical force *F* on the other end. We use a Cartesian coordinate system x_i where origin *O* is the centre of the sample cross section at the wall, the direction x_1 is the axis of the sample, x_2 is the upward vertical and $Ox_1x_2x_3$ forms a direct orthogonal axis system, i.e. the rotation around Ox_3 from Ox_1 to Ox_2 advances the corkscrew in the positive direction on Ox_3. The objective of this study is to investigate the conditions of crack propagation in the sample under load. The specimen is assumed large, homogeneous, isotropic, deforming elastically and fracturing without plasticity.

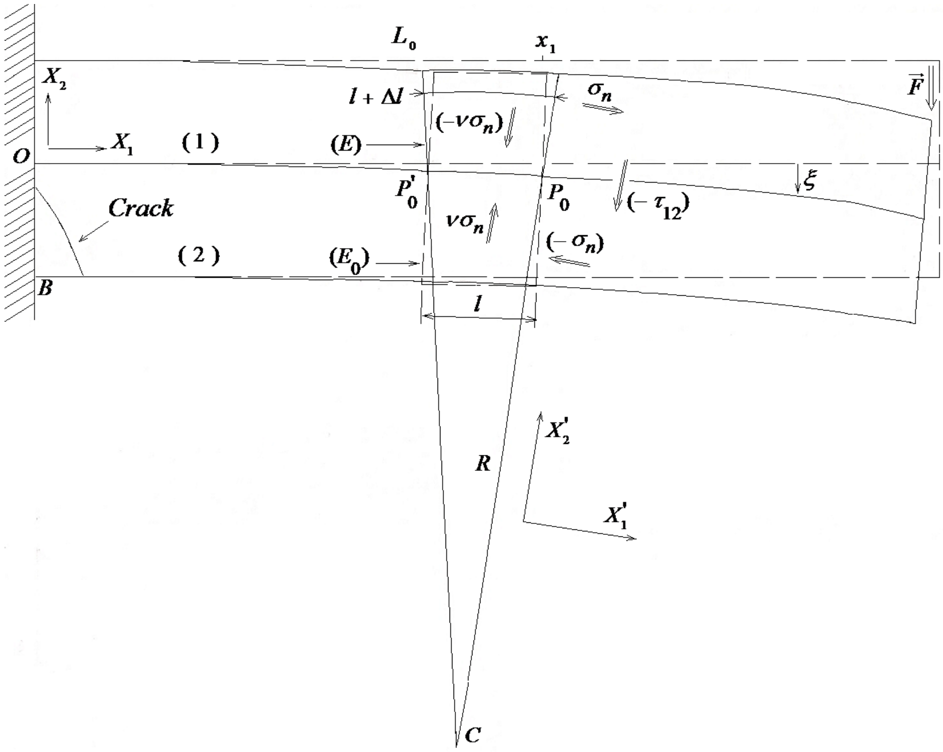

Figure 1 : *Rectangular bar (length L_0, thickness e_0 and width l_0 , measured along x_1, x_2 and x_3, respectively) under bending by a terminal transverse force F. Regions* (1) *and* (2) *are in tension and compression, respectively; these are separated by the neutral line. The specimen is dashed before bending and solid under load. Internal stresses are represented about a point P_0 on the neutral line with coordinate x_1: these are the tension and compression* $(\pm\sigma_n)$ *, The associated induced Poisson* $(\pm\nu\sigma_n)$ *, and the shear* $(-\tau_{12})$ *. The other symbols are defined in the text*

The elastic deformation in the bar under load at equilibrium ***(Figure 1)*** has been well studied [1] and the subject is presented in courses on the elastic properties of solids (bachelor cycle at the university) [2].

The upper part (1) in the bar lengthens, the lower portion (2) shrinks in length. There is no length variation in the median plane (plane separating parts (1) and (2) before loading); this plane is called the "neutral plane". As a result, region (1) is in tension and (2) in compression. Consider a portion of the specimen ***(Figure 1)*** located between the points P_0' and P_0 on the neutral line. Its shape (E_0) is dashed before loading with a length *l*. In the deformed state, its solid-line shape (*E*) is a portion of a circular ring whose faces at points P_0' and P_0 meet at a point C, centre of the ring. We define a Cartesian system $P_0\, x_1' x_2' x_3$ where $P_0 x_1'$ is perpendicular to CP_0 with a positive direction directed towards the end of the bar under load, $P_0 x_2'$ is the axis CP_0 directed upward and $P_0 x_3$ is parallel to Ox_3 through P_0 . The length $(l + \Delta l)$ of (*E*) along x_1' depends on the value x_2' of the position where the measurement is performed; we have

$$\frac{\Delta l}{l} = \frac{x_2'}{R} \tag{1}$$

where $R = CP_0$. *The corresponding internal stress is*

$$\sigma_{11}' = E\,\frac{x_2'}{R} \tag{2}$$

where E is Young's modulus. The magnitude of σ_{11}' is denoted by σ_n ($\left|\sigma_{11}'\right| \equiv \sigma_n$). σ_{11}' and x_2' have the equal sign, positive in part (1) of (*E*) and negative in part (2); hence (*E*) suffers tension and compression forces in regions (1) and (2), respectively, both parallel to $P_0 x_1'$. The force associated to the stresses on (*E*) about P_0 is equal to *F*. This means that tangential forces are present that are directed along $P_0 x_2'$ (in the negative direction), to which corresponds a shear stress σ_{12}' sensibly equal to [2]

$$\sigma_{12}' = -F / S_0 \equiv -\tau_{12} \tag{3}$$

The magnitude of the resultant moment *M* of the forces at P_0 is given by

$$M = \int_{Section} x_2' \sigma_n\, ds = \frac{E}{R} \int_{Section} x_2'^{\,2}\, ds \equiv \frac{EI}{R} \tag{4}$$

where $I = \int x_2^{'2} ds$ is the geometric moment of inertia of sample section at P_0 about $P_0 x_3$. Writing that M is equal to the moment (at P_0) $F(L_0 - x_1)$ of the forces acting on the complementary part $(L_0 - x_1)$ of specimen, we obtain

$$\frac{1}{R} = \frac{F(L_0 - x_1)}{EI}; \tag{5}$$

we can then write

$$\sigma'_{11} = \frac{F(L_0 - x_1)}{I} x'_2 = \tau_{12} \frac{S_0(L_0 - x_1)}{I} x'_2 \tag{6}$$

The equilibrium shape of the bar is extracted from

$$\frac{1}{R} = \frac{\partial^2 \xi / \partial x_1^2}{\left[1 + (\partial \xi / \partial x_1)^2\right]^{3/2}} \cong \frac{\partial^2 \xi}{\partial x_1^2} \tag{7}$$

for small displacements ξ ***(Figure 1)*** of points of the bar along x_2. Integrating (7) gives

$$\xi(x_1) = \frac{F}{EI}\left(\frac{L_0 x_1^2}{2} - \frac{x_1^3}{6}\right) \tag{8}$$

with boundary conditions $\xi(x_1 = 0) = 0$ and $\partial \xi / \partial x_1 (x_1 = 0) = 0$.

How crack behaves under load in the specimen? Assume that a crack expands from an arbitrary x_1 position inside the material. From the stresses about $P_0(x_1, \xi(x_1), x_3)$ ***(Figure 1)***, one can distinguish two distinct types of crack propagation behaviour, in tension region (1) and compression region (2). In the former zone (1), the crack is essentially loaded in tension σ_n. From experiments, it is well-known that a crack loaded in tension, after fracture propagation over large distances, is directed perpendicularly to the applied tension. In this position, Poisson's contraction $(-\nu\sigma_n)$ produces no relative displacement of the faces of the crack, hence does not contribute to crack motion (as commented earlier [3]). However, taking into account the shear $(-\tau_{12})$ (much smaller than σ_n), the crack may deviate from its natural position by a small angle.

In the latter region (2), the crack suffers mainly a compression $(-\sigma_n)$ ***(Figure 1)***. It is also well-known that a crack, experiencing a compressive stress, aligns itself parallel to the compression direction (see [3] and references therein). In this orientation, an opening of the crack is performed by Poisson's tension $\nu\sigma_n$. A deviation of the crack (due to $(-\tau_{12})$) from the compression direction is also to be taken into account. The crack extension force (per unit length of the crack front) G_1 at the tip of the crack that moves in region (1) is much larger than G_2 the corresponding quantity in part (2) of the specimen. Hence, this is under the condition $G_2 = 2\gamma$ (where γ is the surface energy) that complete failure will take place. The crack model used to calculate G_2 is described in Section 2. Physical quantities associated with that modelling (stresses about the crack tip, crack extension force and natural orientation of the crack tip, for instance) form Section 3. The properties of fracture (crack path and failure stresses) are described and discussed in Section 4. Section 5 gives a conclusion to this study.

II - MODELLING METHODOLOGY

It is planned to calculate a crack extension force G_2 pertaining to the tip a crack that is moving downward (region (2) in ***Figure 1***). The considered model is presented in ***Figure* 2**. This is an infinitely extended elastic medium. With respect to a Cartesian system $O'x_1'x_2'x_3'$, the crack with half-length *l* extends along x_1' from $x_1' = -a$ to *a*; it is inclined by an acute angle θ_2 with respect to $O'x_1'x_3'$ with a straight front running indefinitely in the x_3'- direction. The acting stresses are : (1) a uniform compression $(-\sigma_n)$ acting in the x_1'-direction, (2) a uniform Poisson's tension $\nu\sigma_n$ in the x_2'-direction and (3) a uniform shear $(-\tau_{12})$ lying in $x_2'x_3'$ and parallel to x_2'. The crack is represented by two families

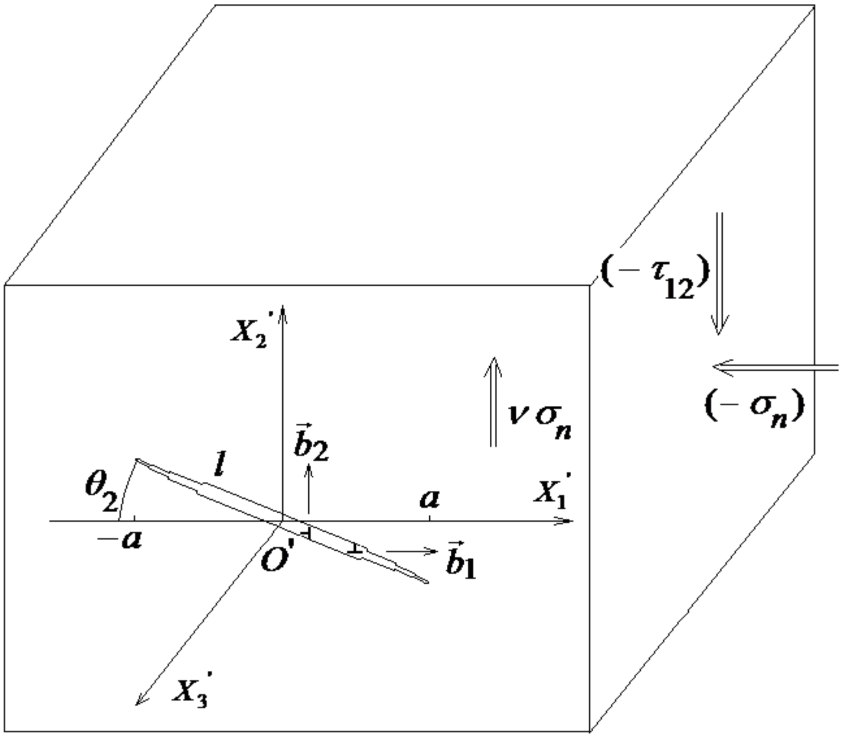

Figure 2 : *Inclined crack by angle* θ_2 *with respect to* $O'x_1'x_3'$*, with a straight front parallel to* $O'x_3'$*. Crack dislocations* (1 *and* 2) *are straight edges parallel to* x_3' *with* $\vec{b}_1$ *along the applied compression* x_1' *- direction and* $\vec{b}_2$ *along the induced Poisson's tension* x_2' *-direction. A shear* $(-\tau_{12})$ *acts in* $x_2'x_3'$ *planes in the* x_2' *- direction*

(1) and (2) of straight edge dislocations parallel to x_3' with infinitesimal Burgers vectors $\vec{b}_1 = (b,0,0)$ and $\vec{b}_2 = (0,b,0)$, respectively. The two families are assumed to be continuously distributed over the crack area from $x_1' = -a$ to a. The dislocation distribution function $D_i(x_1')$ (i= 1 and 2) gives the number of dislocations i in a small interval dx_1' about x_1' as $D_i(x_1')dx_1'$. The correspondence with ***Figure 1*** is that O' can be viewed as located in region (2) or about P_0 and the directions x_i' have the same meaning (Section 1); O' and the stresses depend on the position x_1 along the bar. The tensor (σ^A) of the applied stresses in $O'x_1'x_2'x_3'$ reads

$$(\sigma^{A}) = \begin{pmatrix} -\sigma_n & -\tau_{12} & 0 \\ -\tau_{12} & \nu\sigma_n & 0 \\ 0 & 0 & \nu\sigma_n \end{pmatrix} \tag{9}$$

For a dislocation with Burgers vector $(b,0,0)$ lying indefinitely in the $x_3^{'}$ – direction and displaced by $x_2^{'} = h$ from the origin, the stress field is given at $\vec{x}' = (x_1^{'}, x_2^{'}, x_3^{'})$ by

$$\sigma_{12}^{(1)}(\vec{x}') = C\,\frac{x_1^{'}\left(x_1^{'2} - (x_2^{'} - h)^2\right)}{r^4},\quad \sigma_{11}^{(1)}(\vec{x}') = -C\,\frac{(x_2^{'} - h)\left(3x_1^{'2} + (x_2^{'} - h)^2\right)}{r^4},$$

$$\sigma_{22}^{(1)}(\vec{x}') = C\,\frac{(x_2^{'} - h)\left(x_1^{'2} - (x_2^{'} - h)^2\right)}{r^4},\quad \sigma_{33}^{(1)}(\vec{x}') = -2C\nu\,\frac{(x_2^{'} - h)}{r^2},$$

$$\sigma_{13}^{(1)} = \sigma_{23}^{(1)} = 0 \tag{10}$$

$r^2 = x_1^{'2} + (x_2^{'} - h)^2$ and $C = \mu b / 2\pi(1-\nu)$ where μ is the shear modulus. For a dislocation with Burgers vector $(0,b,0)$ lying along $x_3^{'}$ and displaced by $x_2^{'} = h$ from the origin, the stress is

$$\sigma_{12}^{(2)}(\vec{x}') = C\,\frac{(x_2^{'} - h)(x_1^{'2} - (x_2^{'} - h)^2)}{r^4};$$

$$\sigma_{ii}^{(2)}(\vec{x}') = C\,\frac{x_1^{'}}{r^2}\left(\frac{x_1^{'2}(\delta_{i1} + \delta_{i2}) + (x_2^{'} - h)^2(-\delta_{i1} + 3\delta_{i2})}{r^2} + 2\nu\delta_{i3}\right),\ i = 1,\ 2 \text{ and } 3;$$

$$\sigma_{j3}^{(2)}(\vec{x}') = 0,\ j = 1 \text{ and } 2; \tag{11}$$

δ_{ij} *is the Kronecker delta.*

The crack analysis requires the determination of the equilibrium crack dislocation distribution D_i, relative displacement of the faces of the crack, crack-tip stress and crack extension force. Any point on the crack is given by $P_s = (x_1^{'}, x_2^{'} = -px_1^{'}, x_3^{'})$ with $|x_1^{'}| < a$ and $p = \tan\theta_2 > 0$. We ask the crack faces to be free of traction, this gives

$$\begin{cases} \overline{\sigma}_{12} + p\overline{\sigma}_{11} = 0 \\ \overline{\sigma}_{22} + p\overline{\sigma}_{21} = 0 \\ \overline{\sigma}_{32} + p\overline{\sigma}_{31} = 0 \end{cases} \tag{12}$$

$\overline{\sigma}_{ij}$ stands for the total stress at any point (x_1', x_2', x_3') in the medium and is linked to D_i; in (12), we are only concerned with the points of the crack faces. $\overline{\sigma}_{ij}$ is written as

$$\overline{\sigma}_{ij} = \sigma_{ij}^{A} + \overline{\sigma}_{ij}^{(1)} + \overline{\sigma}_{ij}^{(2)} \tag{13}$$

where σ_{ij}^{A} are the *ij*- elements of matrix (σ^{A}) (9) and

$$\overline{\sigma}_{ij}^{(n)}(x_1', x_2', x_3') = \int_{-a}^{a} \sigma_{ij}^{(n)}(x_1' - x, x_2', x_3') D_n(x)dx \quad (n = 1 \text{ and } 2) \tag{14}$$

here $\sigma_{ij}^{(n)}$ (n = 1or 2) is the stress field produced by a dislocation displaced by $x_2' = h = -px$ from the origin with Burgers vector $(b,0,0)$ or $(0,b,0)$ ((10) and (11)). Introducing the expressions for $\overline{\sigma}_{ij}$ in (12) , we obtain

$$\begin{cases} -(\tau_{12} + p\sigma_n) + C\displaystyle\int_{-a}^{a} \frac{D_1(x)dx}{x_1' - x} = 0 \\ \nu\sigma_n - p\tau_{12} + C\displaystyle\int_{-a}^{a} \frac{D_2(x)dx}{x_1' - x} = 0 \end{cases} \tag{15}$$

(15) forms the governing equations of our modelling. The corresponding solutions are well-known

$$D_1(x_1') = -\frac{\tau_{12} + p\sigma_n}{\pi C} \frac{x_1'}{\sqrt{a^2 - x_1'^2}},$$
$$D_2(x_1') = \frac{\nu\sigma_n - p\tau_{12}}{\pi C} \frac{x_1'}{\sqrt{a^2 - x_1'^2}} \tag{16}$$

with relative displacement ϕ_i of the faces of the crack in the x_1' (i= 1) and x_2' (i= 2) directions

$$\phi_1(x_1') = -\frac{b(\tau_{12} + p\sigma_n)}{\pi C}\sqrt{a^2 - x_1'^2},$$
$$\phi_2(x_1') = \frac{b(\nu\sigma_n - p\tau_{12})}{\pi C}\sqrt{a^2 - x_1'^2} \quad (17)$$

III - CALCULATION RESULTS

Below are presented expressions for the stress about the crack tip and crack extension force G_2. We shall look for crack configuration at which G_2 is maximum. This configuration would correspond to the natural configuration of the propagating crack; this is under this configuration that we shall apply the condition for crack propagation $G_2 = 2\gamma$. The stresses ahead of the crack tip at point $P_s = (x_1', x_2' = -px_1', x_3')$, $x_1' = a + s$, $0 < s \ll a$, are given, for sufficiently small values of s, by the following relation :

$$\overline{\sigma}_{ij}(s) = \sum_{n=1}^{2} \int_{a-\delta a}^{a} \sigma_{ij}^{(n)}(x_1' - x, -px_1', x_3') D_n(x) dx \; ; \; x_1' = a + s \quad (18)$$

with $\delta a \ll a$. Restricting ourselves to the stresses involved in the calculation of the crack extension force, we obtain

$$\overline{\sigma}_{11}(s) = \frac{p(3 + p^2)K_1^* + (1 - p^2)K_2^*}{(1 + p^2)^2}\frac{1}{\sqrt{2\pi}\sqrt{s}},$$
$$\overline{\sigma}_{12}(s) = \frac{(1 - p^2)(K_1^* - pK_2^*)}{(1 + p^2)^2}\frac{1}{\sqrt{2\pi}\sqrt{s}},$$
$$\overline{\sigma}_{11}(s) = \frac{-p(1 - p^2)K_1^* + (1 + 3p^2)K_2^*}{(1 + p^2)^2}\frac{1}{\sqrt{2\pi}\sqrt{s}} \quad (19)$$

where $K_1^* = -(\tau_{12} + p\sigma_n)\sqrt{a\pi}$ and $K_2^* = (\nu\sigma_n - p\tau_{12})\sqrt{a\pi}$; terms with K_i^* are due to dislocation family i (i=1 and 2).

The crack extension force can be calculated following a procedure described in [4]; we have also referred to this in a number of works [3,5]. This gives G_2 at $P_a = (x_1' = a, x_2' = -pa, x_3')$ as

$$G_2(P_a) = \frac{(1-\nu^2)l\pi}{E(1+p^2)}\left((\tau_{12} + p\sigma_n)^2 + (\nu\sigma_n - p\tau_{12})^2\right) \equiv G_a \qquad (20)$$

The behaviour of G_a as a function of *p* is as follows. G_a is minimum at *p*= 0, increases up to a maximum at $p = p_+$ and then decreases asymptotically to a value $G_a(\infty)$ as *p* tends to infinity; we have

$$p_+ = \frac{1}{2}\left((1+\nu)\overline{\sigma}_n + \sqrt{(1+\nu)^2\overline{\sigma}_n^2 + 4}\right),$$

$$G_a(p=0) = \frac{(1-\nu^2)l\pi\tau_{12}^2}{E}\left(1+\nu^2\overline{\sigma}_n^2\right),$$

$$G_a(p=p_+) = \frac{(1-\nu^2)l\pi\tau_{12}^2}{E}\left(1+\overline{\sigma}_n^2 + \frac{2(1-\nu)\overline{\sigma}_n}{(1+\nu)\overline{\sigma}_n + \sqrt{(1+\nu)^2\overline{\sigma}_n^2 + 4}}\right) \equiv G_a^{\max},$$

$$G_a(\infty) = \frac{(1-\nu^2)l\pi\tau_{12}^2}{E}\left(1+\overline{\sigma}_n^2\right) \qquad (21)$$

where $\overline{\sigma}_n = \sigma_n / \tau_{12}$. The fracture stress $\tau_{12} \equiv \sigma_f$ is obtained from $G_a^{\max} = 2\gamma$ as

$$\sigma_f = \sigma_t \sqrt{\frac{(1+\nu)\overline{\sigma}_n + \sqrt{(1+\nu)^2\overline{\sigma}_n^2 + 4}}{(1+\nu)\overline{\sigma}_n^3 + (3-\nu)\overline{\sigma}_n + (1+\overline{\sigma}_n^2)\sqrt{(1+\nu)^2\overline{\sigma}_n^2 + 4}}} \qquad (22)$$

Where

$$\sigma_t = \sqrt{\frac{2\gamma E}{\pi(1-\nu^2)l}} \qquad (23)$$

IV - DISCUSSION

We shall assume that σ_n in our modelling ***(Figure 2)*** is given by the magnitude of $\sigma_{11}^{'}$ (6) that depends on the x_1 - position along the specimen and value of $x_2^{'}$ along the axis CP_0 ***(Figure 1)***. As a consequence, the various physical quantities derived in Section 3 depend on x_1 and $x_2^{'}$. Under such conditions, what is the failure stress σ_F in the bending of fracture specimens by terminal transverse load?

Fracture may well occur at different x_1 - positions along the length of the specimen in ***Figure 1***, due to pre-existing flaw for instance. Hence for fixed x_1, we may assume first (*a*) that σ_F is given by (22) with highest σ_n value corresponding to $x_2^{'} = -e_0/2$. This corresponds to position $B = (x_1 = 0, x_2 = -e_0/2)$ in ***Figure 1*** for $x_1 = 0$. We have $\overline{\sigma}_n(B) = 6L_0/e_0$ (for $I = l_0 e_0^3/12$) that is generally much larger than unity. We first discuss the behaviour of the crack about *B*. The crack inclination angle θ_2^+ with respect to the $x_1^{'}$ - direction ***(Figure 2)*** (given by $p_+ = \tan\theta_2^+$ (21)) takes about *B* for $\overline{\sigma}_n >> 1$, the simple form $\tan\theta_2^+(B) = (1+\nu)\overline{\sigma}_n(B) = 6(1+\nu)L_0/e_0$. With $L_0/e_0 = 10$, this gives $\theta_2^+(B) = 89°$ that corresponds to a crack path not far from the x_2 – direction. Using (6) with $x_1 = 0$, $x_2^{'} = x_2$ and writing

$dx_2/dx_1 = -(1+\nu)\overline{\sigma}_n = -(1+\nu)S_0L_0|x_2|/I$,

we obtain by integration

$$x_2 = x_2(0)e^{(1+\nu)V_0x_1/I} \tag{24}$$

where $x_2(0) < 0$ in compression region (2) and $V_0 = L_0S_0$ is the volume of the specimen. (24) indicates that the crack path is in the form of an exponential about the wall in region (2). A complete crack shape about the wall may be a path parallel to Ox_2 for $x_2 > 0$ and in the form of an exponential for $x_2 < 0$ with a beginning point somewhere below *O* (as shown schematically in ***Figure 1***). A pre-existing flaw may promote failure out of the wall; again, the exponential path of the crack would be preserved there. The fracture stress σ_f (22) in bending by terminal load *F* ***(Figure 1)*** is proportional to the quantity σ_t (23). This latter expression is nothing else than the stress at failure when a specimen with a crack of half-length *l* is loaded in tension. When $\overline{\sigma}_n >> 1$, (22) reduces to

$$\frac{\sigma_f}{\sigma_t} = \frac{1}{\overline{\sigma}_n} = \frac{e_0}{6L_0} \tag{25}$$

and in term of the bending moment $M \equiv M_f$,

$$M_f = \frac{2I\sigma_t}{e_0} \tag{26}$$

It is important to stress that these relations are intimately related to the existence of a shear stress τ_{12}. In presence of a shear stress, G_a (20) has a maximum from which is deduced σ_f (22); however, in absence of a shear stress ($\tau_{12} = 0$) as we shall discuss, G_a has no maximum with p and (22) (hence 25, 26) does not exist. This brings us to discuss the particular case of bending without shear stress. An example corresponds to the bending of a bar by terminal couple [1]; the bending moment $M \equiv M_c$ (4) is constant along the bar so that $\sigma'_{11} = M_c x'_2 / I$ (6). The crack extension force G_{2c} in this situation is given by (20) with $\tau_{12} = 0$. We have

$$G_{2c}(P_a) = \frac{(1-\nu^2)\sigma_n^2 l\pi}{E}\left(\frac{\nu^2 + p^2}{1 + p^2}\right) \tag{27}$$

This is the expression of the crack extension force at the tip of an inclined crack under compression and internal Poisson tension discussed by [3]. The factor $(\nu^2 + p^2)/(1 + p^2)$ in (27) increases continuously with p from ν^2 (p= 0) to a value limited by 1 (p large). In absence of shear stress, the inclined crack ***(Figure 2)*** tends to align parallel to the applied compression $(-\sigma_n)$ direction (see [3] and references therein). The natural configuration of the crack system corresponds to p= 0. Under such conditions, the relation $G_{2c} = 2\gamma$ yields

$$\sigma_n = \frac{1}{\nu}\sigma_t \tag{28}$$

Taking $\sigma_n = M_c e_0 / 2I$, we have

$$M_c = \frac{2I\sigma_t}{\nu e_0} \tag{29}$$

To compare with (26), we take $\nu = 1/3$ for isotropic materials and obtain $M_c = 3M_f$; this indicates that three times larger bending moment is required to fracture a specimen in bending by terminal couple. Other consideration (b) for σ_F is to average (22) over specimen section; this yields

$$\sigma_F = \frac{2\sigma_t}{e_0} \int_0^{e_0/2} \sqrt{\frac{(1+\nu)\overline{\sigma}_n + \sqrt{(1+\nu)^2 \overline{\sigma}_n^2 + 4}}{(1+\nu)\overline{\sigma}_n^3 + (3-\nu)\overline{\sigma}_n + (1+\overline{\sigma}_n^2)\sqrt{(1+\nu)^2 \overline{\sigma}_n^2 + 4}}}\, dx_2' \quad (30)$$

Doing the same thing with (28) gives a couple

$$M_C = \frac{4I\sigma_t}{\nu e_0} \quad (31)$$

that is twice the value (29). Considerations (*a*) and (*b*) above demand confrontation with experiments that will be the subject of separate works.

The present modelling assumes a crack path perpendicular to the applied tension direction in tension regions of the fractured specimen and parallel to the applied compression in compression regions. This means that fracture specimens must have large dimensions in order that crack propagation over macroscopic distance is the controlling mechanism of failure. Crack nucleation would be internal. Fracture under boundary effects so frequently associated to small specimens may not be described by the present study.

V - CONCLUSION

In the present study, an analysis has been performed of the conditions of failure in a rectangular bar, horizontally in a wall at one end and under load about the other end, either by terminal transverse load or terminal couple. The specimen consists of two distinct parts (1) and (2) separated by a non-deformed (neutral) plane. Region (1) suffers tension and (2) compression. It is known from experiment that a crack propagating over macroscopic distances under tension, tends to align perpendicularly to the applied tension direction, whilst under a compressive stress, the equal crack would be directed along the compression direction. In the last orientation, this is the induced Poisson's tension , much smaller than the applied stress that is responsible for the opening of the crack faces. We are thus lead to consider two distinct crack extension forces G_1 and G_2 pertaining to both regions. Because G_1 is much larger than G_2, this is under the condition $G_2 = 2\gamma$ that complete failure would take place. We then proceed to a calculation of G_2, using a model of a finite crack in an infinitely extended homogeneous medium, that permits us to describe a natural configuration of the crack under load and a relation between the failure stress and the dimensions of the specimen.

This study allows to distinguish two types of applied loading : (1) flexion associated with shear (case of terminal transverse load) and (2) pure bending without shear (case of terminal couple). Three times larger bending moment at least is required to fracture the specimen in the latter applied loading condition.

REFERENCES

[1] - A. E. H. LOVE, "A treatise on the mathematical theory of elasticity", Dover Publications, Inc., New York, 4th ed. (1927).

[2] - J. BUTTET, " Cours de physique générale I. Propriétés élastiques des solides et des fluides", Département de physique (EPFL), Ecublens (1981).

[3] - P. N. B. ANONGBA, J. BONNEVILLE and A. JOULAIN, *Rev. Ivoir. Sci. Technol.*, 17 (2011) 37 – 53.

[4] - B. A. BILBY and J. D. ESHELBY, In: "Fracture", Ed. Academic Press (H. Liebowitz), New York, Vol 1 (1968) 99 – 182.

[5] - P. N. B. ANONGBA, *Rev. Ivoir. Sci. Technol.*, 16 (2010) 11 – 50