BRITTLE CRACKS UNDER COMPRESSION : INTRODUCING POISSON EFFECT

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ABSTRACT

An estimate is made of the contribution of Poisson effect to the conditions for the propagation of a planar crack subjected to uniform compression in the framework of linear isotropic elasticity. When comparison is made with tension, $(1/\nu)$ times larger stress is required to break a fracture specimen in compression (ν is Poisson's ratio). The treatment considers an inclined planar crack with respect to the applied compression direction and provides an expression of the crack extension force G per unit length of the crack front as a function of the inclination slope p of the crack. A representation of the crack by a continuous distribution of edge dislocations with infinitesimal Burgers vectors is adopted. It is shown that the inclined crack can be described by two distinct dislocation families responding to the applied compression and the induced internal Poisson tension, respectively.

Keywords : *continuum mechanics, crack propagation, dislocations, Poisson effect, fracture mechanisms.*

RÉSUMÉ

Fissures fragiles sous compression : introduction de l'effet Poisson

Une estimation est faite de la contribution de l'effet Poisson aux conditions de propagation d'une fissure plane soumise à une compression uniforme dans le cadre de l'élasticité linéaire. En termes de contrainte et lorsqu'une comparaison est faite avec la tension, il est $(1/\nu)$ fois plus difficile de casser une éprouvette d'essai en compression (ν est le rapport de Poisson). L'analyse considère une fissure inclinée par rapport à la direction de la compression et donne une expression de la force d'extension *G* de la fissure par unité de longueur du front de fissure en fonction de la pente d'inclinaison *p*. La représentation d'une fissure par une distribution continue de dislocations avec des vecteurs de Burgers infinitésimaux est adoptée. Il est montré que la fissure inclinée peut être décrite par deux familles distinctes de dislocation obéissant à la compression appliquée et à la tension de Poisson induite, respectivement.

Mots-clés : *mécanique des milieux continus, propagation de fissure, dislocations, effet Poisson, mécanismes de rupture.*

I - INTRODUCTION

Consider as an illustration in ***Figure 1*** a rectangular fracture specimen with large dimensions, isotropic and elastic. Initially the specimen is dashed. With respect to a Cartesian coordinate system x_i and under the action of uniform compressive stress $(-\sigma_a)$ along the x_1-direction, the specimen shrinks in the x_1-direction but extends according to Poisson in the x_2 and x_3 directions; the shape then taken by the specimen is in solid (***Figure 1***). To the specimen is attached a planar crack with finite dimensions along x_1 and x_2 but infinite in the calculations (below) in the x_3-direction. The crack is symmetrical with respect to Ox_3, inclined by an angle θ with respect to Ox_1x_3 and has a straight front along x_3. When the crack is in Ox_1x_3 ($\theta = 0$), considering the applied compression $(-\sigma_a)$ only, we show in the following that the crack extension force *G* per unit length of the crack front is zero. However this result is in conflict with numerous experimental observations revealing that the crack is able to propagate axially along x_1. We shall come back to the experimental evidences in Section 4. At this point, we only stress that in addition to an extension along x_1, there is an opening of the crack along x_2. Things happen as if the crack was subjected to a tension in the x_2-direction. A possible origin of an induced tension is the well-known Poisson effect which, in the framework of linear elasticity obeying Hooke's law, leads to a stress $(\nu\sigma_a)$ directed along x_2 (***Figure 1***) where ν is Poisson's ratio. The goal of the present study is to give the conditions for the propagation of the crack depicted schematically in ***Figure 1*** taking account of Poisson effect. The physical quantity to be provided by the analysis is the crack extension force *G* as a function of the inclination angle θ.

In the present study, the crack is represented by a continuous distribution of infinitesimal dislocations. The stress field induced by the crack in the surrounding medium will be given by the dislocations of the distribution. This method for describing the crack is well understood from the general work by Bilby and Eshelby [1]. We stress below that two distinct couples of edge dislocations (see Section 3) lead to the same value of the crack extension force, a value that is in full agreement with stress intensity factors given in the literature (Sih et al. [2]; Sih and Liebowitz [3]) in the case of an inclined planar crack with the same geometry as in ***Figure 1*** but under tension along x_1.

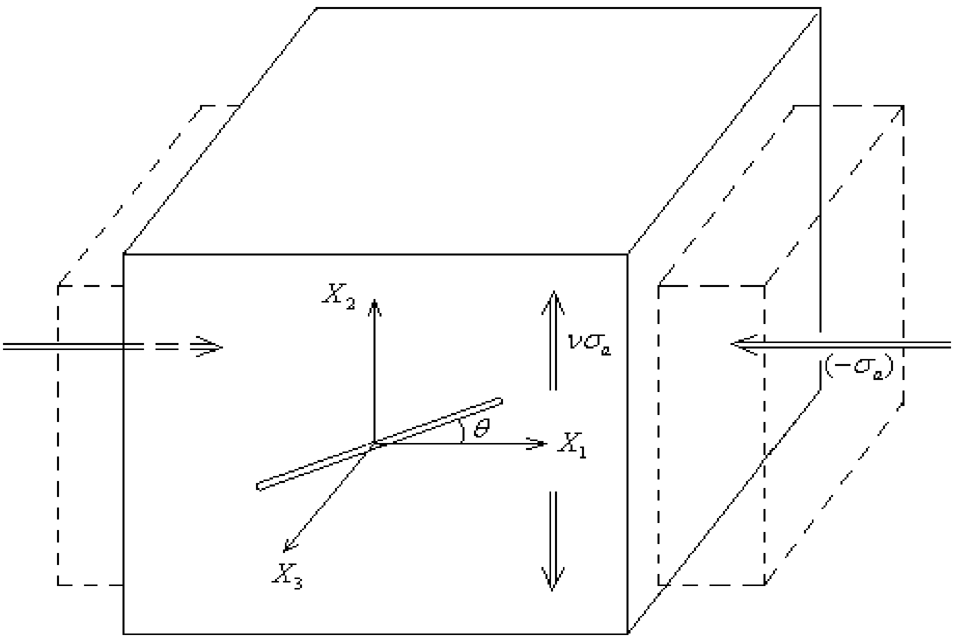

Figure 1 : *Fracture specimen under uniform applied compression* $(-\sigma_a)$*,* $\sigma_a > 0$*, in the* x_1 *– direction. The specimen is dashed before compression and solid under loading. An inclined planar crack with a straight front parallel to* x_3 *is associated to the body as well as an assumed uniform internal tension* $(\nu\sigma_a)$ *in the* x_2 *– direction corresponding to the Poisson effect.*

In Section 2, we report the stresses of the edge dislocations used in the crack analysis as well as the assumed stress field of Poisson effect. Section 3 details the crack analysis. In Section 4, a discussion is made of the contribution of Poisson effect in the understanding of brittle fracture in compression.

II - POISSON EFFECT AND DISLOCATION STRESS FIELDS

In the situation of ***Figure 1*** where the specimen is loaded in compression in the x_1 – direction, Poisson effect corresponds to a strain $\varepsilon_{22}^{(P)}$ in the x_2 – direction and $\varepsilon_{33}^{(P)}$ in the x_3 – direction. The effect of $\varepsilon_{22}^{(P)}$ corresponds to an opening of the crack when $\theta = 0$. Poisson law reads

$$\varepsilon_{22}^{(P)} = -\nu\varepsilon_{11}^{a} \quad (1)$$

where ε_{11}^{a} is the strain along x_1 that results from the compression $(-\sigma_a)$. $(-\sigma_a) = E\varepsilon_{11}^{a}$ from Hooke's law and E is Young's modulus. To $\varepsilon_{22}^{(P)}$ is associated an internal stress $\sigma_{22}^{(P)}$ using Hooke's law, this leads to

$$\sigma_{22}^{(P)} = \nu\sigma_a . \quad (2)$$

We assume an internal uniform Poisson stress. Similarly, Poisson tension $\sigma_{33}^{(P)}$ in the x_3 – direction reads $\sigma_{33}^{(P)} = \nu\sigma_a$ so that the stress tensor corresponding to Poisson in the basis $(\vec{e}_1, \vec{e}_2, \vec{e}_3)$ associated with the directions x_i is given as

$$\left(\sigma^{(P)}\right) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \nu\sigma_a & 0 \\ 0 & 0 & \nu\sigma_a \end{pmatrix} . \quad (3)$$

We shall deal with two types of edges parallel to x_3 arranged on an inclined crack plane. The crack dislocations would then have an elevation h with respect to the horizontal Ox_1x_3 plane. For a dislocation with Burgers vector $(b,0,0)$ lying indefinitely in the x_3 – direction and displaced by $x_2 = h$ from the origin, the stress field is given at $\vec{x} = (x_1, x_2, x_3)$ by

$$\sigma_{12}^{(1)}(\vec{x}) = C\,\frac{x_1\left(x_1^2-(x_2-h)^2\right)}{r^4}, \qquad \sigma_{11}^{(1)}(\vec{x}) = -C\,\frac{(x_2-h)\left(3x_1^2+(x_2-h)^2\right)}{r^4},$$
$$\sigma_{22}^{(1)}(\vec{x}) = C\,\frac{(x_2-h)\left(x_1^2-(x_2-h)^2\right)}{r^4}, \quad \sigma_{33}^{(1)}(\vec{x}) = -2C\nu\,\frac{(x_2-h)}{r^2},$$
$$\sigma_{13}^{(1)} = \sigma_{23}^{(1)} = 0; \tag{4}$$

$r^2 = x_1^2 + (x_2-h)^2$ and $C = \mu b/2\pi(1-\nu)$ where μ is the shear modulus. For a dislocation with Burgers vector $(0,b,0)$ lying along x_3 and displaced by $x_2 = h$ from the origin, the stress is

$$\sigma_{12}^{(2)}(\vec{x}) = C\,\frac{(x_2-h)(x_1^2-(x_2-h)^2)}{r^4};$$
$$\sigma_{ii}^{(2)}(\vec{x}) = C\,\frac{x_1}{r^2}\left(\frac{x_1^2(\delta_{i1}+\delta_{i2})+(x_2-h)^2(-\delta_{i1}+3\delta_{i2})}{r^2}+2\nu\delta_{i3}\right),\ i = 1, 2 \text{ and } 3;$$
$$\sigma_{j3}^{(2)}(\vec{x}) = 0,\ j = 1 \text{ and } 2; \tag{5}$$

δ_{ij} is the Kronecker delta.

III - ANALYSIS OF CRACKS

The crack analysis will be performed with two different dislocation arrangements corresponding to ***Figures 2*** and ***3***. For each arrangement we shall successively give the equilibrium dislocation distributions, the crack-tip stresses and the crack extension force.

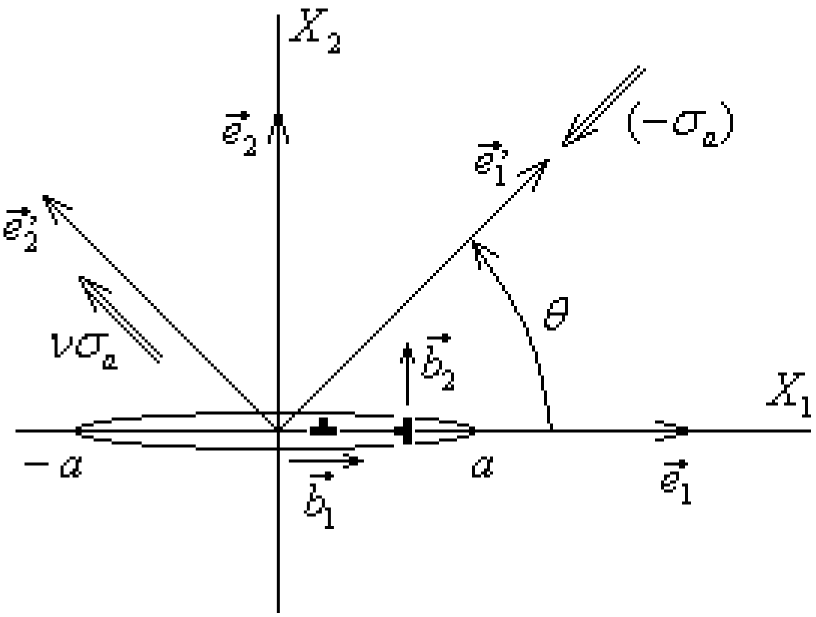

Figure 2 : *Crack dislocations geometry corresponding to straight edges parallel to* x_3 *with* $\vec{b}_1$ *along* x_1 *in the crack plane and* $\vec{b}_2$ *along* x_2 *perpendicular to crack. Uniform applied compression* $(-\sigma_a)$ *acts along* $\vec{e}_1'$ *and internal Poisson* $(\nu\sigma_a)$ *along* $\vec{e}_2'$. *Note that in this arrangement* $\vec{b}_1$ *and* $\vec{b}_2$ *are not directed along the acting forces.*

Consider first the crack dislocations description given in ***Figure 2***. The medium is infinitely extended with a planar crack (straight front) in the Ox_1x_3 plane extending from $x_1 = -a$ to a and running indefinitely in the x_3-direction. We consider two families of straight edge dislocations in the crack. Families 1 and 2 are both parallel to x_3 with Burgers vectors $\vec{b}_1 = (b,0,0)$ and $\vec{b}_2 = (0,b,0)$ in the x_1 and x_2 directions respectively. The two dislocation families are assumed to be continuously distributed over the crack area between $x_1 = -a$ to a. The system is subjected to uniform applied compression $(-\sigma_a)$ at infinity in the $\vec{e}_1'$ direction (***Figure 2***). The basis $(\vec{e}_1', \vec{e}_2')$ is obtained from $(\vec{e}_1, \vec{e}_2)$ after a rotation about Ox_3 by an angle θ. Furthermore we assume uniform Poisson tension $\nu\sigma_a$ in the $\vec{e}_2'$ direction. The dislocation distribution function $D_i(x_1)$ ($i = 1$ and 2 for edges 1 and 2) gives the number of dislocations i in a small interval dx_1 about x_1 as

$D_i(x_1)dx_1$. We are concerned with the problem of finding the equilibrium distributions D_i of the dislocations under the combined action of their mutual repulsions and the force exerted on them by $(-\sigma_a)$ and $\nu\sigma_a$. We ask the crack faces to be traction free, this gives

$$\begin{cases} \bar{\sigma}_{12} = 0 \\ \bar{\sigma}_{22} = 0 \\ \bar{\sigma}_{23} = 0 \end{cases}. \qquad (6)$$

$\bar{\sigma}_{ij}$ stands for the total stress at any point (x_1, x_2, x_3) in the medium and is linked to D_i; in (6), we are only concerned with the points of the crack faces. $\bar{\sigma}_{ij}$ is written as

$$\bar{\sigma}_{ij} = \sigma_{ij}^{a} + \sigma_{ij}^{(P)} + \bar{\sigma}_{ij}^{(1)} + \bar{\sigma}_{ij}^{(2)} \qquad (7)$$

where σ_{ij}^{a} corresponds to the applied compression, $\sigma_{ij}^{(P)}$ to Poisson and

$$\bar{\sigma}_{ij}^{(n)}(x_1, x_2, x_3) = \int_{-a}^{a} \sigma_{ij}^{(n)}(x_1 - x_1^{'}, x_2, x_3) D_n(x_1^{'}) dx_1^{'} \quad (n = 1 \text{ and } 2); \qquad (8)$$

here $\sigma_{ij}^{(n)}$ (n = 1or 2) is the stress field produced by a dislocation displaced by $x_2 = h$ ($h = 0$ in the case of ***Figure 2***) from the origin with Burgers vector $(b,0,0)$ or $(0,b,0)$ ((4) and (5)). With respect to the system $(O, \vec{e}_1, \vec{e}_2, \vec{e}_3)$ we have

$$\left(\sigma^{a}\right) = (-\sigma_a) \begin{pmatrix} \cos^2\theta & \sin 2\theta/2 & 0 \\ \sin 2\theta/2 & \sin^2\theta & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\left(\sigma^{(P)}\right) = \nu\sigma_a \begin{pmatrix} \sin^2\theta & -\sin 2\theta/2 & 0 \\ -\sin 2\theta/2 & \cos^2\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}; \qquad (9)$$

$\sigma_{ij}^{(n)}$ in (8) may be taken from (4) and (5) with $h=0$. The traction free boundary condition (6) then leads to the following system of integral equations

$$\begin{cases} (-\sigma_a)\dfrac{(1+\nu)p}{1+p^2} + C\displaystyle\int_{-a}^{a}\dfrac{D_1(x_1')dx_1'}{x_1 - x_1'} = 0 \\ (-\sigma_a)\left(\dfrac{p^2-\nu}{1+p^2}\right) + C\displaystyle\int_{-a}^{a}\dfrac{D_2(x_1')}{x_1 - x_1'}dx_1' = 0 \end{cases} \tag{10}$$

where $p=\tan\theta$. The Cauchy principal values of the integrals are to be taken. The type of solution is well known [1]:

$$D_1(x_1) = \frac{(1+\nu)p}{1+p^2}\frac{(-\sigma_a)}{\pi C}\frac{x_1}{\sqrt{a^2-x_1^2}},$$
$$D_2(x_1) = \left(\frac{p^2-\nu}{1+p^2}\right)\frac{(-\sigma_a)}{\pi C}\frac{x_1}{\sqrt{a^2-x_1^2}}. \tag{11}$$

The corresponding relative displacement ϕ_i of the crack faces, in the $x_1 (i=1)$ and x_2 $(i=2)$ directions, are:

$$\phi_1(x_1) = \frac{(1+\nu)p}{1+p^2}(-\sigma_a b/\pi C)(a^2-x_1^2)^{1/2},$$
$$\phi_2(x_1) = \left(\frac{p^2-\nu}{1+p^2}\right)(-\sigma_a b/\pi C)(a^2-x_1^2)^{1/2}. \tag{12}$$

D_i is unbounded at $x_1 = \pm a$ and the ϕ_i curve vertical at these end points.

We are now interested in stress values in the neighbourhood of the crack tip located at $x_1 = a$ at point P with coordinates $(x_1, x_2 = 0, x_3)$. Substituting $x_1 = a+s$, $0 < s << a$, $\bar{\sigma}_{ij}$ (7) is given by the following formula:

$$\bar{\sigma}_{ij}(s) = \sum_{n=1}^{2}\int_{a-\delta a}^{a}\sigma_{ij}^{(n)}(a+s-x_1', x_2=0, x_3)D_n(x_1')dx_1' \tag{13}$$

with $\delta a << a$. This stress expression means that only those dislocations located about the crack front in x_1-interval $[a-\delta a\ ,\ a]$ will contribute significantly to the stress at $x_1 = a + s$ ahead of the crack tip as *s* tends to zero; any other contribution will be negligible for a sufficiently small value of *s*. We observe that this formula is precise with no place for any other kind of additional stress term. Restricting ourselves to $\bar{\sigma}_{12}$ and $\bar{\sigma}_{22}$ since they are involved in the calculation of the crack extension force *G* (see in the following) we obtain

$$\bar{\sigma}_{12}(s) = \frac{K_1}{\sqrt{2\pi}} \frac{1}{\sqrt{s}}, \qquad \bar{\sigma}_{22}(s) = \frac{K_2}{\sqrt{2\pi}} \frac{1}{\sqrt{s}} \tag{14}$$

where

$$K_1 = \frac{(1+\nu)p}{1+p^2}(-\sigma_a)\sqrt{a\pi}, \qquad K_2 = \frac{p^2 - \nu}{1+p^2}(-\sigma_a)\sqrt{a\pi}. \tag{15}$$

By removing ν in (15), we recover the stress intensity factors displaced by Sih et al. [2] (see also [3]) for an inclined planar crack loaded in tension.

We proceed to calculate the crack extension force following Bilby and Eshelby [1]. This procedure is detailed since it is also used in the analysis of the crack of ***Figure 3***. Allow the right-hand tip of the crack in ***Figure 2*** to advance (say rigidly for simplicity) from $x_1 = a$ to $a + \delta a$, but apply forces to the freshly formed surfaces to prevent relative displacement; the energy of the system is unaltered. Now allow these forces to relax to zero so that the crack extends effectively from a to $a + \delta a$. The work done by these forces corresponds to a decrease of the energy of the system which we shall estimate (the energy of the system consists of the elastic energy of the medium and the energy of the loading mechanism). The element $ds = dx_3 dx_1$ of the fracture plane ahead of the crack front, at a point $P = (x_1, x_2 = 0, x_3)$, may be defined by $d\vec{s} = \vec{\gamma} ds$ where $\vec{\gamma}$ is the unit vector perpendicular to ds pointing to the positive $x_2 -$direction. We obtain $d\vec{s} = (0,1,0) dx_1 dx_3$. The relevant component of the force acting on ds in the $x_i -$direction is $\bar{\sigma}_{ij} ds_j$ (the summation convention on repeated subscripts applies) where $\bar{\sigma}_{ij}$ are stresses ahead of the shorter crack; thus the energy change associated with ds is $\bar{\sigma}_{ij} ds_j \Delta u^{(i)} / 2$ (here a summation is also considered over $i = 1$ and 2) where $\Delta u^{(i)}$ is the difference in displacement across the lengthened crack, just behind its tip, in the $x_i -$direction. When the crack advances from

$x_1 = a$ to $a + \delta a$, the energy decrease associated with a surface element $dx_3 \delta a$ is

$$-\delta E = dx_3 \left(\frac{1}{2} \int_a^{a+\delta a} \bar{\sigma}_{12} \Delta u^{(1)} dx_1 + \frac{1}{2} \int_a^{a+\delta a} \bar{\sigma}_{22} \Delta u^{(2)} dx_1 \right). \quad (16)$$

Let *G* be a derivative of the energy of the system with respect to crack area. *G* corresponds to the limiting value taken by $-\delta E / dx_3 \delta a$ as δa decreases to zero: $G = \lim_{\delta a \to 0} -\delta E / dx_3 \delta a$. Stresses $\bar{\sigma}_{ij}$ generally consist of terms that are either bounded or unbounded as x_1 tends to a; only those stress terms that are singular may contribute a non-zero value to *G*; the bounded terms all contribute nothing. Hence we can use (14) for $\bar{\sigma}_{12}$ and $\bar{\sigma}_{22}$. $\Delta u^{(i)}$ may be obtained from the solution of (10) modified to allow for the fact that the crack extends from $x_1 = -a$ to $a + \delta a$ instead of from $-a$ to *a*. We may use (12) for $\Delta u^{(i)}$ following Bilby and Eshelby [1]; this leads to

$$\begin{aligned} G(P_0) &= \left(K_1^2 + K_2^2\right)\left(1 - \nu^2\right)/E \\ &= \left(\frac{\nu^2 + p^2}{1 + p^2}\right) a \pi \sigma_a^2 \left(1 - \nu^2\right)/E. \end{aligned} \quad (17)$$

Expression (17) gives the value of *G* at an arbitrary point $P_0(a, x_2 = 0, x_3)$ along the front of the planar crack with half length *a*. *G* is defined as the crack extension force per unit edge length of the crack front [1].

Consider again the inclined planar crack problem under uniform compression but with a different type of crack dislocations, ***Figure 3***. The main difference with ***Figure 2*** is that the Burgers vectors of the dislocations are now directed along the applied compression $(-\sigma_a)$ x_1-direction and Poisson $(\nu\sigma_a)$ x_2-direction, but are not linked to the plane of the crack. Here *a* designates the projected half length *l* of the crack along x_1.

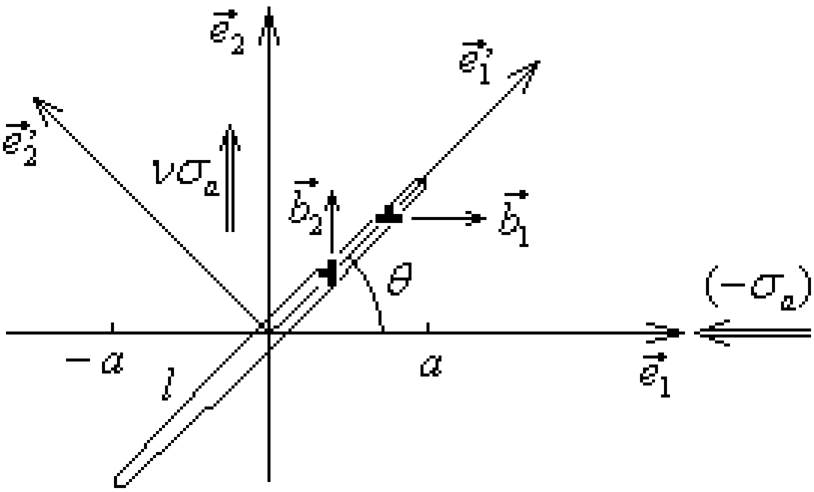

Figure 3: *Crack dislocations arrangement corresponding to straight edges parallel to x_3 with $\vec{b}_1$ along x_1 out of the crack plane and $\vec{b}_2$ along x_2 non perpendicular to crack. Uniform applied compression $(-\sigma_a)$ acts along $\vec{e}_1$ and internal Poisson $(\nu\sigma_a)$ along $\vec{e}_2$. The projected half length of the crack along x_1 is a. In this geometry $\vec{b}_1$ and $\vec{b}_2$ are directed along the acting forces. We also have $a = l\cos\theta$ where l is the half length of the inclined crack.*

The traction free boundary condition at an arbitrary point $P = (x_1, x_2 = px_1, x_3)$, $|x_1| < a$, of the faces of the crack reads

$$\begin{cases} \bar{\sigma}_{12} - p\bar{\sigma}_{11} = 0 \\ \bar{\sigma}_{22} - p\bar{\sigma}_{12} = 0 \\ \bar{\sigma}_{23} - p\bar{\sigma}_{13} = 0 \end{cases} . \quad (18)$$

Again for $\bar{\sigma}_{ij}$ we use (7): σ_{ij}^{a} and $\sigma_{ij}^{(P)}$ are obvious; for $\bar{\sigma}_{ij}^{(n)}$ we need to express $\sigma_{ij}^{(n)}$ ((4) and (5)) paying attention to the fact that *h* in (4) and (5) is position dependent in the dislocation distribution. Introducing the expressions for $\bar{\sigma}_{ij}$ in (18) we obtain

$$\begin{cases} p\sigma_a + C\int\limits_{-a}^{a}\dfrac{D_1(x_1')dx_1'}{x_1 - x_1'} = 0 \\ \nu\sigma_a + C\int\limits_{-a}^{a}\dfrac{D_2(x_1')}{x_1 - x_1'}dx_1' = 0 \end{cases} . \tag{19}$$

The corresponding solutions are

$$D_1^*(x_1) = \frac{p\sigma_a}{\pi C}\frac{x_1}{\sqrt{a^2 - x_1^2}},$$
$$D_2^*(x_1) = \frac{\nu\sigma_a}{\pi C}\frac{x_1}{\sqrt{a^2 - x_1^2}} \tag{20}$$

with relative displacement ϕ_i^* of the faces of the crack, in the x_1 ($i = 1$) and x_2 ($i = 2$) directions

$$\phi_1^*(x_1) = (p\sigma_a b/\pi C)(a^2 - x_1^2)^{1/2}$$
$$\phi_2^*(x_1) = (\nu\sigma_a b/\pi C)(a^2 - x_1^2)^{1/2}. \tag{21}$$

It is seen from (20) that in absence of Poisson effect $D_2^* = 0$ and in absence of the applied compression $D_1^* = 0$. Hence dislocation family 1 responds to the applied compression and family 2 to Poisson.

Using (13) and paying attention to put $x_2 = px_1$ in $\sigma_{ij}^{(n)}$, the stress $\overline{\sigma}_{ij}$ at a point $P = (x_1, x_2 = px_1, x_3)$ ahead of the crack front ($x_1 = a + s$, $0 < s << a$) may be calculated. We only give $\overline{\sigma}_{11}$, $\overline{\sigma}_{12}$ and $\overline{\sigma}_{22}$ (to be used for the calculation of G):

$$\overline{\sigma}_{11} = \frac{1}{(1+p^2)^2}\left(-p(3+p^2)K_1^* + (1-p^2)K_2^*\right)\frac{1}{\sqrt{2\pi}\sqrt{s}},$$
$$\overline{\sigma}_{12} = \frac{1-p^2}{(1+p^2)^2}\left(K_1^* + pK_2^*\right)\frac{1}{\sqrt{2\pi}\sqrt{s}},$$
$$\overline{\sigma}_{22} = \frac{1}{(1+p^2)^2}\left(p(1-p^2)K_1^* + (1+3p^2)K_2^*\right)\frac{1}{\sqrt{2\pi}\sqrt{s}} \tag{22}$$

where $K_1^* = p\sigma_a\sqrt{a\pi}$ and $K_2^* = \nu\sigma_a\sqrt{a\pi}$; terms with K_i^* are due to dislocation family i ($i = 1$ and 2).

To calculate G, consider that the crack tip at $x_1 = a$ and elevation $x_2 = pa$ (see ***Figure 3***) advances from $x_1 = a$ to $a + \delta a$ on the inclined crack plane. We have to attach to a surface element Δs ahead of the crack front an energy decrease ($-\delta E$). We take $\Delta s = \int_a^{a+\delta a} ds$ with $ds = \sqrt{1+p^2}\, dx_1 dx_3$; this gives $\Delta s = \sqrt{1+p^2}\,\delta a dx_3$. The unit vector $\vec{\gamma}$ perpendicular to ds is $\vec{\gamma} = 1/\sqrt{1+p^2}\,(-p,1,0)$ so that $d\vec{s} = \vec{\gamma} ds = (-p,1,0) dx_1 dx_3$. We have $(-\delta E) = \frac{1}{2}\int_a^{a+\delta a}\sum_i (\bar{\sigma}_{i1} ds_1 + \bar{\sigma}_{i2} ds_2)\Delta u^{(i)}$, the integration being performed with respect to x_1. Making use of (22) for $\bar{\sigma}_{ij}$ and (21) for $\Delta u^{(i)}$ we arrive at

$$\begin{aligned} G(P_0) &= \lim_{\delta a \to 0} -\delta E / \Delta s \\ &= \frac{1}{\sqrt{1+p^2}}\left(K_1^{*2} + K_2^{*2}\right)\left(1-\nu^2\right)/E \\ &= \left(\frac{\nu^2 + p^2}{1+p^2}\right) l \pi \sigma_a^2 \left(1-\nu^2\right)/E \end{aligned} \qquad (23)$$

that is identical to (17); $P_0 = (a, x_2 = pa, x_3)$. The two different ways for describing the crack dislocations (***Figures 2*** and ***3***) lead to the same expression of the crack extension force. The factor $(\nu^2 + p^2)/(1+p^2)$ in (23) increases continuously with p from the value ν^2 ($p = 0$) to a value limited by 1 (p large). It is interesting to mention from (23) that G consists of two separate terms, with K_i^* ($i = 1$ and 2), associated with the dislocation families 1 and 2 (***Figure 3***) respectively.

The arrangement of the crack dislocations given in ***Figure 3*** is convenient. Consider a fracture specimen that breaks under general loading along a fracture surface (this may be a non-planar surface) corresponding to the loading conditions. One may decompose the applied loading into a tension σ_{22}^a (mode I) along x_2 (use ***Figure 1*** to illustrate), a shear σ_{12}^a (mode II) along x_1 and a shear σ_{23}^a (mode III) along x_3. Hence it will be possible to represent the crack by families i of dislocations with Burgers vectors $\vec{b}_i$ along x_i ($i = 1$, 2 and 3) responding essentially to the associated mode i (II , I and III respectively) ; see for example Anongba [4, 5].

IV - DISCUSSION AND CONCLUSION

We first observe that Poisson effect plays no role during the propagation process when the crack is loaded in tension. Indeed, if the crack of ***Figure 1*** is assumed to be located in Ox_1x_3 ($p = 0$) and loaded in tension along x_2, Poisson effect would correspond to a lateral contraction of the medium parallel to x_1 and x_3; under such conditions there is no relative displacement of the faces of the crack associated with these contractions. We conclude that Poisson effect plays no role in tension. In contrast, under applied compression $(-\sigma_a)$ along x_1, ***Figure 1***, Poisson tension $(\nu\sigma_a)$ along x_2 opens a crack located in Ox_1x_3 thus improving the conditions for crack motion.
Again, consider the specimen in ***Figure 1*** with $p = 0$. In absence of the Poisson effect, we obtain $G = 0$ from (17) or (23): this result is in conflict with experimental observations (see below). When the fracture specimen breaks at a stress level σ_c, using the condition $G = 2\gamma$ where γ is the surface energy, we obtain

$$\sigma_c = \frac{1}{\nu}\sqrt{\frac{2\gamma E}{\pi(1-\nu^2)l}}\,. \tag{24}$$

The same specimen with half length *l* loaded in tension along x_2 (***Figure 1***, $p = 0$) would fracture at a stress level σ_t given by

$$\sigma_t = \sqrt{\frac{2\gamma E}{\pi(1-\nu^2)l}}\,. \tag{25}$$

We arrive at

$$\sigma_c = \frac{1}{\nu}\sigma_t\,. \tag{26}$$

Assuming $\nu = 1/3$ (isotropic medium), this gives $\sigma_c = 3\sigma_t$ suggesting that 3 times larger stress is needed to fracture a specimen in compression as compared to tension.

There is a considerable amount of experimental observations showing cracks propagating axially along x_1 under compression. We may refer, among others, to the following works: Fig. 14.1 in Paul [6]; Figs. 6.9.1(a) and 6.9.2(a) in Nemat-Nasser and Hori [7]; Fig. 4 in Laplanche et al. [8] and Fig. V-63 in Giacometti [9]. These cracks display the following features (refer to ***Figure 1*** with $\theta = 0$ for the $x_i -$directions):

- Crack propagation occurs in absence of plasticity.
- The crack length may be of the order of ½ mm and even larger.
- Larger cracks are more opened. In other words, there is a gradual opening of the faces of the crack during crack extension. The opening occurs in the $x_2 -$direction perpendicular to the crack Ox_1x_3 plane.

It appears necessary to assume a tension in the $x_2 -$direction to be able to explain axial propagation of the crack along x_1 under compression. The present study gives the contribution of the Poisson effect to the conditions for crack propagation under compression in the case of an isotropic homogeneous elastic body.

It is interesting to discuss Poisson effect in connection with experimental measurements of the stress to fracture in compression. In most available works (see, for example, [10 to 13]), specimens contain a starter crack in the form of a slit oriented at an acute angle $\theta \neq 0$ with respect to the $x_1 -$direction of compression as illustrated in ***Figure 1***. Unfortunately fracture of the specimen under load does not correspond to an extension of the starter crack in its initial inclination direction, a necessary condition for result (23) to apply. In contrast, a new crack initiates from the tip of the slit and propagates in a completely different direction so as to become gradually parallel to the $x_1 -$axis of compression. Under such conditions it appears necessary that the starter crack be parallel to the $x_1 -$axis ($\theta = 0$), a situation that is uncommonly reported.

In the experiments by Ashby and Hallam [13] in PMMA plates, the smallest reported slit inclination angle is $\theta = 15°$. This is because as θ decreases to zero, the stress to fracture the specimen increases and falls in a range corresponding to crack propagation in presence of plasticity. Ashby and Hallam [13] quoted with insistence that PMMA is not an ideally brittle solid and displayed accordingly the associated stress-strain curve (their Fig. 10). The conditions we shall take from their work (see Fig. 4 in [13]) correspond to lowest fracture stress levels (we seek linear-elastic deformation): $\theta = 15°$ in our notation; $l = 8$ mm starter crack half length; σ_c (stress to fracture) given by them in the form of a ratio $R = \sigma_c \sqrt{\pi l} / K_{IC} = 6.5$ with $K_{IC} = 1.0$ MPa m$^{1/2}$; K_{IC} is the fracture toughness of the PMMA for slow crack growth taken from [14]. Introducing Poisson effect using (23) with

$p = 0$ approximately, $\nu\sigma_c\sqrt{\pi l}$ is the appropriate quantity to be compared with K_{IC}, this gives a ratio $R^{(P)} = \nu\sigma_c\sqrt{\pi l} / K_{IC} = 2.5$ ($\nu = 0.38$, PMMA). We would expect $R^{(P)} = 1$ if Poisson effect operates exclusively in the crack propagation process along x_1. The observed discrepancy may have different possible origins. First the starter crack is $\theta = 15°$ inclined from the compression $x_1 -$axis and should actually be perfectly aligned along x_1. We may also invoke the facts that the slit is not perfectly sharp or a localized plasticity exists at the tip of the crack. We should equally be aware that Poisson $x_2 -$extension may be inhibited in appreciable parts of the fracture specimen adjacent to the grips. Ashby and Hallam [13] noted that the behaviour of the growing crack is extremely sensitive to the condition of the ends of the sample. In summary, under uniform applied compression $(-\sigma_a)$ along x_1 (***Figure 1***), an axial $x_1 -$propagation of a planar straight edge crack in Ox_1x_3 requires a non-zero relative displacement of the faces of the crack in the $x_2 -$direction. This may be provided by Poisson effect and requires $(1/\nu)$ times larger stress when compared with tensile fracture experiments.

When a pre-existing crack in the form of a slit inclined by an angle $\theta \neq 0$ is loaded in compression in the geometry of ***Figure 1***, as mentioned above, further extension of the flaw does not occur in the initial inclination direction. What occurs is the nucleation of a new crack from the tip of the slit that deviates appreciably from the initial inclination direction (refer to [15]). The new crack gradually extends in a stable manner with increasing axial compression, curving toward an orientation parallel to the compression $x_1 -$direction. Several theoretical and experimental analyses have been undertaken to understand this complex process (see among others [12, 13, 16, 17]). This is not the aim of the present study to give a detailed account of the implications of Poisson effect into these theories. What is obvious is that when some $x_2 -$tension of the order of Poisson is introduced into the displayed results, then the crack growth becomes unstable after a short extension of the newly created crack, leading to $x_1 -$axial splitting (see: Fig. 16 and 18 in [13]; Fig. 4 in [17]). In contrast, in absence of a tension along x_2 the new crack grows to a finite length and then stops inside the medium: this reveals the importance of $x_2 -$tension (independent of crack morphology) in the complete splitting process. Agreement is achieved between theory and experiment as indicated by Ashby and Hallam [13] (see the ir Fig. 18).

In conclusion, Poisson effect is potentially able to account for the splitting (parallel to the compression direction) of fracture specimens observed in real materials.

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